Polynomial Approximation on Real-Analytic Varieties in $\mathbf{C}^n$

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Abstract

We prove: *Let Σ be a compact real-analytic variety in an open set $\Omega \subset \mathbf{C}^n$. Assume (i) Σ is polynomially convex and (ii) every point of Σ is a peak point for $P(\Sigma)$. Then $P(\Sigma) = C(\Sigma)$.* This generalizes a previous result of the authors on polynomial approximation on three-dimensional real-analytic submanifolds of $\mathbf{C}^n$.

1. Introduction

We consider the problem of approximating arbitrary continuous functions on a compact subset X of n-dimensional complex Euclidean space $\mathbf{C}^n$ by polynomials in the coordinate functions $z_1, \ldots, z_n$. We let $C(X)$ denote the space of all continuous complex-valued functions on X, with norm $\|g\|_X = \max\{|g(z)| : z \in X\}$, and we let $P(X)$ denote the closure of the set of polynomials in $C(X)$. The polynomially convex hull of X will be denoted $\widehat{X}$. That is,

$$\widehat{X} = \{z \in \mathbf{C}^n : |Q(z)| \leq \|Q\|_X \text{ for every polynomial } Q\}$$

Two necessary conditions for $P(X) = C(X)$ are:

(i) X is polynomially convex, i.e. $X = \widehat{X}$;

(ii) Every point of X is a peak point for $P(X)$, i.e., given $p \in X$, there exists $f \in P(X)$ with $f(p) = 1$ and $|f| < 1$ on $X \setminus \{p\}$.

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With a general uniform algebra A on a compact metric space X replacing $P(X)$, and with (i) replaced by the condition that the maximal ideal space of A coincides with X, it was once conjectured that together these two necessary conditions for $A = C(X)$ were also sufficient to imply $A = C(X)$. However, a counterexample to this "peak-point conjecture" was produced by Brian Cole in his 1968 thesis (see the appendix to [5], or [12], section 19). Additional counterexamples to the peak point conjecture have been given in the context of polynomial and rational approximation in several complex variables. In particular these counterexamples show that conditions (i) and (ii) above are not sufficient for $P(X) = C(X)$. For more on this, see [2], [3], [4] and [9].

In [2] the first two authors established a peak-point result for two-manifolds:

Theorem 1.1 *Let M be a compact two-dimensional differentiable manifold, and A a uniform algebra on M generated by continuously differentiable functions. Assume that the maximal ideal space of A is M, and that each point of M is a peak point for A. Then $A = C(M)$.*

An example of Izzo [9] shows that Theorem 1.1 fails for uniform algebras on smooth three manifolds. However, in [3] the authors established the following:

Theorem 1.2 *Let Σ be a real-analytic three-dimensional submanifold of $\mathbf{C}^n$. Let X be a compact subset of Σ such that ∂X (the boundary of X relative to Σ), is a two-dimensional submanifold of class C^1. If X satisfies conditions (i) and (ii) above, then $P(X) = C(X)$.*

Our purpose in this note is to extend Theorem 1.2 to real-analytic varieties of arbitrary dimension in $\mathbf{C}^n$. We prove:

Theorem 1.3 *Let Σ be a compact real-analytic variety in an open set $\Omega \subset \mathbf{C}^n$. Assume (i) $\Sigma = \widehat{\Sigma}$ and (ii) every point of Σ is a peak point for $P(\Sigma)$. Then $P(\Sigma) = C(\Sigma)$.*

If ∂B_n denotes the boundary of the unit ball in $\mathbf{C}^n$, then every point of ∂B_n is a peak point for polynomials, and so as an immediate consequence of Theorem 1.3 we have the following:

Corollary 1.4 *If $\Sigma \subset \partial B_n$ is a compact polynomially convex real-analytic variety, then $P(\Sigma) = C(\Sigma)$.*

In order to explain the idea of the proof of Theorem 1.3, we recall the proof of Theorem 1.2. The main tool is a result of Hörmander and Wermer [8] (proved by them for sufficently smooth manifolds, and later generalized by O'Farrell, Preskenis and Walsh [11] to C^1 manifolds) that, in essence, reduces questions of approximation on subsets of real submanifolds of $\mathbf{C}^n$ to approximation on the points where the tangent space to the manifold contains a complex line. The following dual formulation suffices (see the discussion following Proposition 2.3 of [3]):

Theorem 1.5 *Let X be a polynomially convex compact subset of $\mathbf{C}^n$, and let X_0 be a compact polynomially convex subset of X such that $X \setminus X_0$ is a totally real submanifold of $\mathbf{C}^n$, of class C^1. If μ is a measure on X such that $\int_X f\, d\mu = 0$ for all $f \in P(X)$, then μ is supported on X_0. In particular, $P(X) = C(X)$ if and only if $P(X_0) = C(X_0)$.*

To establish Theorem 1.2, one shows, under assumptions (i) and (ii), that the set of points in X where Σ has a complex tangent is a real-analytic variety V of dimension at most two. To show that $P(X) = C(X)$, it is enough, by Theorem 1.5, to show that $P(\partial X \cup V) = C(\partial X \cup V)$. Applying Theorem 1.5 again, one reduces the problem to showing that $P(\partial X \cup V^*) = C(\partial X \cup V^*)$, where V^* is the union of the singular set of the variety V together with the set of regular points of V at which V has a complex tangent. Theorem 1.1 implies that $P(\partial X) = C(\partial X)$. Using the peak-point property (ii), V^* can be shown to have two-dimensional Hausdorff measure zero. This suffices to show (see Lemma 3.1 of [3]) that $P(\partial X \cup V^*) = C(\partial X \cup V^*)$, and completes the proof.

The fact that the two-dimensional Hausdorff measure of V^* is zero is essential to the proof we have just described; this allows one to apply the Hartogs-Rosenthal theorem on rational approximation in the plane to certain projections. In attempting to generalize Theorem 1.2 to real-analytic manifolds of dimension greater than three, one would like to proceed in a similar way: reduce the problem of approximation on Σ to approximation on V, where V is the set of points in Σ at which Σ has a complex tangent. Assumptions (i) and (ii) imply, as before, that V is a variety whose dimension is strictly less than the dimension of Σ, and so one hopes to proceed by induction on dimension, repeatedly applying Theorem 1.5 as before, eventually reducing to approximation on a sufficiently small set (two-dimensional Hausdorff measure zero). The fact that at each stage one has to consider approximation on varieties makes it desirable, for purposes of the induction, to prove Theorem 1.3 in the category of varieties, rather than restricting to real-analytic manifolds. However, a problem arises: the singular set of a real-analytic variety (unlike the case of complex-analytic varieties) need not itself be a variety. This problem forces a slightly different approach. We show that locally the union of the set of complex tangent points and the singular set of each variety in question is contained in a variety of smaller dimension, and this allows the induction to proceed.

2. Real-Analytic Varieties

In this section we review the basic facts about real-analytic varieties necessary for the proof of Theorem 1.3.

If U is an open subset of $\mathbf{R}^m$, $C^\omega(U)$ will denote the set of all real-valued functions which are real-analytic in U. If $\mathcal{F}$ is a subset of $C^\omega(U)$, we set

$$V_U(\mathcal{F}) = \{x \in U : f(x) = 0, \ \ \forall f \in \mathcal{F}\}.$$

A subset Σ of an open set $\Omega \subset \mathbf{R}^m$ is said to be a (real-analytic) *variety in* Ω if for each $p \in \Omega$ there exists an open neighborhood $U \subset \Omega$ of p and a finite set $\mathcal{F} \subset C^\omega(U)$ with $\Sigma \cap U = V_U(\mathcal{F})$. It follows that Σ is a relatively closed subset of Ω; we say Σ is a *compact variety in* Ω if it is a compact subset of Ω. The class of varieties is closed under finite unions and intersections.

The *dimension* of a real-analytic variety Σ is the largest natural number d such that for some $p \in \Sigma$, there exists a neighborhood U of p such that $\Sigma \cap U$ is a real-analytic submanifold of U of dimension d, and the set of *regular points* of Σ, denoted Σ_{reg}, is the set of points $p \in \Sigma$ with this property. Σ_{reg} is a relatively open subset of Σ. Its complement, the set of *singular points* of Σ, is denoted Σ_{sing}.

A variety Σ in Ω is said to be *irreducible* if whenever $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_1, Σ_2 varieties in Ω, then either $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$. If Σ is an irreducible variety in Ω, and Σ' is a proper subset of Σ that is also a variety in Ω, then $\dim(\Sigma) > \dim(\Sigma')$ (see [6], Theorem 3.4.8, assertion (15)).

Unlike the case of complex analytic varieties, the singular set of a real-analytic variety need not itself be a variety. Moreover, a real-analytic variety is not necessarily the union of a finite number of irreducible varieties. However, we have the following local properties (see [10], p. 31-42, especially Proposition 5, and [6], section 3.4.10):

(a) If Σ is irreducible, then for each $p \in \Sigma_{sing}$ there exists a neighborhood U of p and a function $\delta \in C^\omega(U)$ not vanishing identically on $\Sigma \cap U$ so that

$$\Sigma_{sing} \cap U \subset V_U(\{\delta\}) \cap \Sigma \equiv \Delta$$

Moreover, if $d = \dim(\Sigma)$, there is a finite set $\mathcal{G} = \{g_1, \ldots, g_{m-d}\} \subset C^\omega(U)$ so that

$$(\Sigma \cap U) \setminus \Delta = V_{U \setminus \Delta}(\mathcal{G})$$

with $dg_1, \ldots, dg_{m-d}$ linearly independent on $U \setminus \Delta$;

(b) If Σ is a variety in Ω, then for each $p \in \Sigma$ there is a neighborhood U of p and a finite number of irreducible varieties $Z_1, \ldots, Z_s$ in U so that $\Sigma \cap U = \cup_{k=1}^s Z_k$;

(c) Let H^r denote r-dimensional Hausdorff measure. If Σ is a variety of dimension d in Ω then for each compact subset X of Ω, $H^d(\Sigma \cap X) < \infty$ and $H^{d-1}(\Sigma_{sing} \cap X) < \infty$.

We now consider real-analytic varieties in an open subset Ω of $\mathbf{C}^n$, identified with $\mathbf{R}^{2n}$. If M is a real submanifold of Ω, of class C^1, we say that M *has a complex tangent at* $p \in M$ if the real tangent space T_pM of M at p contains a nontrivial complex subspace of $\mathbf{C}^n$. This is equivalent to the condition that the restriction to M of any $(m, 0)$ form vanishes at p, where m is the real dimension of M (see [3], Lemma 2.5). The manifold M is said to be *totally real* if it has no complex tangents. If Σ is a variety in Ω, Σ_c will denote the set of points $p \in \Sigma_{reg}$ such that Σ_{reg} has a complex tangent at p. By expressing the vanishing of each $(m, 0)$ form as the vanishing of an $(m \times m)$ minor of the matrix $J = ((\partial z_l / \partial u_k))$, where $(u_1, \ldots, u_m)$ are local real-analytic coordinates on M, we see that Σ_c is a variety in $\Omega \setminus \Sigma_{sing}$. Set $\Sigma^* = \Sigma_{sing} \cup \Sigma_c$. Note that $\Sigma \setminus \Sigma^*$ is a totally real, real-analytic submanifold of $\Omega \setminus \Sigma_{sing}$.

Lemma 2.1 *Let Σ be a d-dimensional variety in an open set $\Omega \subset \mathbf{C}^n$. Assume dim$(\Sigma_c) < d$. Then for each $p \in \Sigma^*$, there exists an open neighborhood $U \subset \Omega$ of p and a variety Y in U with dim$(Y) < d$ and*

$$\Sigma^* \cap U \subset Y \subset \Sigma \cap U$$

Proof: Fix $p \in \Sigma$, take U and $Z_1, \ldots, Z_s$ as in (b). Let $d_j = \dim(Z_j)$, and set $J = \{j : 1 \leq j \leq s \text{ and } d_j = d\}$. For each $j \in J$, we proceed as follows: shrinking U if necessary, using (a) we obtain a function $\delta_j \in C^\omega(U)$, not vanishing identically on $Z_j \cap U$, with

$$(Z_j)_{sing} \subset \Delta_j \equiv V_U(\{\delta_j\}) \cap Z_j$$

and a set $\mathcal{G}^{(j)} \subset C^\omega(U)$ of $2n - d_j$ functions whose differentials are linearly independent on $U \setminus \Delta_j$, so that

$$Z_j \setminus \Delta_j = V_{U \setminus \Delta_j}(\mathcal{G}^{(j)}).$$

Using the functions in $\mathcal{G}^{(j)}$ and the remarks on the complex tangent set preceding the statement of Lemma 2.1, we may construct a family $\Phi^{(j)}$ of functions in $C^\omega(U)$ so that

$$(Z_j)_c = V_{U \setminus \Delta_j}(\Phi^{(j)} \cup \mathcal{G}^{(j)}).$$

Set $X_j = \Delta_j \cup V_U(\Phi^{(j)} \cup \mathcal{G}^{(j)})$. Then X_j is a variety in U containing Z_j^*. Our assumption on $\dim(\Sigma_c)$ together with the irreducibility of Z_j implies $\dim(X_j) < d$. Let X be the union of the X_j, $j \in J$. Let Z be the union of all pairwise intersections of the Z_j, together with the union of all $Z_j, j \notin J$. Set $Y = X \cup Z$. Then $\Sigma^* \cap U \subset Y \subset \Sigma \cap U$, and $\dim(Y) < d$. $\Box$

3. Proof of Theorem 1.3

We let $B(p, r)$ denote the open ball of radius r centered at p. We will use repeatedly the fact (Lemma 2.1 of [3]) that properties (i) and (ii) are inherited by compact subsets of Σ.

Lemma 3.1 *Let Σ be a compact d-dimensional variety in an open set $\Omega \subset \mathbf{C}^n$. Assume that Σ is polynomially convex and that each point of Σ is a peak point for $P(\Sigma)$. Then for each $p \in \Sigma$, there exist arbitrarily small $r > 0$ and varieties $Y_1, \ldots, Y_d$ in $B(p,r)$ so that if $Y_0 = \Sigma \cap B(p,r)$, then for $1 \leq j \leq d$ we have*

(1) $Y_{j-1}^* \subset Y_j \subset Y_{j-1}$;

(2) $\dim(Y_j) \leq d - j$;

Proof: The proof is by induction on j, $1 \leq j \leq d$. Note that the hypotheses on Σ imply that Σ_c has no interior relative to Σ (see [3], Lemma 3.2). Hence $\dim(\Sigma_c) < d$. We choose r_1, Y_1 as follows: in case $p \in \Sigma \setminus \Sigma^*$, take $r_1 > 0$ so that $B(p, r_1) \cap \Sigma^* = \emptyset$, and take $Y_1 = \emptyset$. If $p \in \Sigma^*$, Lemma 2.1 implies that there exists $r_1 > 0$ and a variety Y_1 in $B(p, r_1)$ so that $\dim(Y_1) \leq d - 1$ and

$$B(p, r_1) \cap \Sigma^* \subset Y_1 \subset B(p, r_1) \cap \Sigma$$

Note that in either case, both (1) and (2) hold for $j = 1$. Now assume by induction that for some k with $1 \leq k < d$, $r_1, \ldots, r_k > 0$ and $Y_1, \ldots, Y_k$ have been chosen with $r_1 \geq r_2 \ldots \geq r_k$, and $Y_1, \ldots, Y_k$ varieties in $B(p, r_k)$, so that (1) and (2) hold for all $j \leq k$. We obtain r_{k+1}, Y_{k+1} as follows: if $p \in Y_k \setminus Y_k^*$, take $Y_{k+1} = \emptyset$ and $r_{k+1} = r_k$. If $p \in Y_k^*$, use Lemma 2.1 to produce r_{k+1}, $0 < r_{k+1} \leq r_k$ and a variety Y_{k+1} with

$$B(p, r_{k+1}) \cap Y_k^* \subset Y_{k+1} \subset B(p, r_{k+1}) \cap Y_k$$

and $\dim(Y_{k+1}) < \dim(Y_k) \leq d - k$ (here we again use Lemma 3.2 of [3]). Then (1) and (2) will hold for $j \leq k + 1$. By induction, we can thus choose $r_1, \ldots, r_d > 0$ and $Y_1, \ldots, Y_d$ so that (1) and (2) hold for $j \leq d$; take $r = r_d$. $\Box$

We now turn to the proof of Theorem 1.3. By duality, it suffices to show that any measure μ on Σ with the property that

$$\int_\Sigma g \, d\mu = 0 \tag{1}$$

for all $g \in P(\Sigma)$ must be the zero measure. Fix a measure μ satisfying (1). We will show that for each point $p \in \Sigma$, there is a neighborhood of p lying outside the support of μ. Fix

$p \in \Sigma$, and choose $r > 0$, and $Y_0, \ldots, Y_d$ as constructed in Lemma 3.1. Now there must be a largest number j such that μ is supported on $(\Sigma \setminus B(p,r)) \cup Y_j$. But for $j < d$,

$$[(\Sigma \setminus B(p,r)) \cup Y_j] \ \setminus \ [(\Sigma \setminus B(p,r)) \cup Y_{j+1}] = Y_j \setminus Y_{j+1}$$

is a totally real submanifold of $\mathbf{C}^n$. By Theorem 1.5, if μ is supported on $(\Sigma \setminus B(p,r)) \cup Y_j$, then μ is supported on $(\Sigma \setminus B(p,r)) \cup Y_{j+1}$. By induction, μ is supported on $(\Sigma \setminus B(p,r)) \cup Y_d$. Note that $\dim(Y_d) = 0$, so the variety Y_d is a discrete point set, and therefore countable. Since every point of Σ is a peak point for $P(\Sigma)$, μ cannot have nonzero mass at any single point, and hence $|\mu|(Y_d) = 0$. Thus μ is supported on $\Sigma \setminus B(p,r)$, and the proof is complete. $\square$

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