Rational Approximation on the Unit Sphere in $\mathbf{C}^2$

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Abstract

For X a compact subset of the unit sphere ∂B in $\mathbf{C}^2$, we seek conditions implying that $R(X) = C(X)$. We conjecture an analogue of the Hartogs-Rosenthal theorem on rational approximation in the plane: if $X \subset \partial B$ is rationally convex and the three-dimensional measure of X is zero, then $R(X) = C(X)$. We make several contributions to the study of this conjecture, and establish rational approximation on certain Lipschitz graphs lying in ∂B. In section 3, we study algebras on certain plane sets with application to approximation on ∂B. In section 4, we weaken the Lipschitz condition, used in section 2, to a Hölder condition.

1. Introduction

For a compact set $X \subset \mathbf{C}^n$, we denote by $R(X)$ the closure in $C(X)$ of the set of rational functions holomorphic in a neighborhood of X. We are interested in finding conditions on X that imply that $R(X) = C(X)$, i.e. that each continuous function on X is the uniform limit of a sequence of rational functions holomorphic in a neighborhood of X.

When $n = 1$, the theory of rational approximation is well developed. Examples of sets without interior for which $R(X) \neq C(X)$ are well-known, the "Swiss cheese" being a prime example. On the other hand, the Hartogs-Rosenthal theorem states that if the two-dimensional Lebesgue measure of X is zero, then $R(X) = C(X)$.

In higher dimensions, there is an obstruction to rational approximation that does not appear in the plane. For $X \subset \mathbf{C}^n$, we denote by $\widehat{X}_r$ the rationally convex hull of X,

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2000 *Mathematics Subject Classification.* Primary 32E30, Secondary 46J10.

which can be defined as the set of points $z \in \mathbf{C}^n$ such that every polynomial Q with $Q(z) = 0$ vanishes at some point of X. The condition $X = \widehat{X}_r$ (X is rationally convex) is both necessary for rational approximation and difficult to establish, in practice, when $n > 1$; in the plane, every compact set is rationally convex.

We will consider primarily subsets of the unit sphere ∂B in $\mathbf{C}^2$. We have been motivated by a desire to obtain an analogue of the Hartogs-Rosenthal theorem in this setting. R. Basener [5] has given examples of rationally convex sets $X \subset \partial B$ for which $R(X) \neq C(X)$; his examples have the form $\{(z, w) \in \partial B : z \in E\}$, where $E \subset \mathbf{C}$ is a suitable Swiss cheese. These sets have the property that $\sigma(X) > 0$, where σ is three-dimensional Hausdorff measure on ∂B. It is reasonable to conjecture that if X is rationally convex, and $\sigma(X) = 0$, then $R(X) = C(X)$. This paper contains several contributions to the study of this question.

In the second section we employ a construction of Henkin [10]. For a measure μ supported on ∂B orthogonal to polynomials, Henkin produced a function $K_\mu \in L^1(d\sigma)$, satisying $\overline{\partial}_b K_\mu = -4\pi^2\mu$. Lee and Wermer established that if $X \subset \partial B$ is rationally convex, and $\mu \in R(X)^\perp$ (i.e., $\int g \; d\mu = 0$ for all $g \in R(X)$), then K_μ extends holomorphically to the unit ball. We show that if the extension belongs to the Hardy space $H^1(B)$, then μ must be the zero measure. Under an assumption on the size of the rational hull of small tubular neighborhoods of X, which we call the *hull-neighborhood property*, we are able to show that K_μ satisfies a certain boundedness condition (see Lemma 2.4 below). From this we deduce (in the proof of Theorem 2.5 below) that $K_\mu \in H^1(B)$ if X is a subset of a Lipschitz graph lying in ∂B. Thus in this case the only measure $\mu \in R(X)^\perp$ is the zero measure, and so $R(X) = C(X)$. In section 4 we show how the same result can be established for graphs of Hölder functions. Also in section 2, we give an example of a class of sets satisfying the hull-neighborhood property.

In the third section we study the algebra generated by $R(E)$ and a smooth function f on a plane set E, and show that if this algebra has maximal ideal space E but does not contain all continuous functions on E, then there is a subset E_0 of E on which $f \in R(E_0)$ and $R(E_0) \neq C(E_0)$. We then use this result to establish rational approximation on certain graphs lying in ∂B.

We use the following notation in addition to that already introduced: B will denote the unit ball in $\mathbf{C}^2$, coordinates of points in $\mathbf{C}^2$ will either be denoted using subscripts, such as $z = (z_1, z_2)$ or by $p = (z, w)$, according to the context. π will denote projection to the first coordinate, i.e. $\pi(z, w) = z$. If z, ζ are points in $\mathbf{C}^2$, $\langle z, \zeta \rangle$ will denote the usual Hermitian inner product of z and ζ.

2. Rational Approximation and the Henkin transform

A basic tool of approximation theory in the plane is the Cauchy transform $\hat{\mu}$ of a measure μ. If μ is a finite complex measure with compact support,

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}.$$

The Cauchy transform $\hat{\mu}(z)$ is integrable with respect to Lebesgue measure m on the plane, is analytic in z off the support of μ, and satisfies the fundamental relation

$$\frac{\partial \hat{\mu}}{\partial \overline{z}} = -\pi \mu$$

in the sense of distributions, i.e.,

$$\int \phi \, d\mu = \frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial \phi}{\partial \overline{z}} \, \hat{\mu} \, dm. \tag{1}$$

In [10], Henkin studied global solutions to the inhomogeneous tangential Cauchy-Riemann equations on the boundary of strictly convex domains in $\mathbf{C}^n$. His work produced transforms analogous in certain respects to the Cauchy transform. In the particular case which concerns us, the boundary of the unit ball in $\mathbf{C}^2$, Henkin introduced the kernel

$$H(z, \zeta) = \frac{\langle Tz, \zeta \rangle}{|1 - \langle z, \zeta \rangle|^2}$$

where $Tz = (\overline{z_2}, -\overline{z_1})$. Given a measure μ supported on a set $X \subset \partial B$, the Henkin transform of μ is defined by

$$K_\mu(z) = \int_X H(z, \zeta) d\mu(\zeta).$$

Henkin showed that the integral defining K_μ converges σ-a.e on ∂B, K_μ is integrable with respect to $d\sigma$ on ∂B, and is smooth on $\partial B \setminus X$. Further, if μ satisfies the condition

$$\int_X P \, d\mu = 0, \qquad \forall \text{ polynomials } P \tag{2}$$

then K_μ satisfies

$$\overline{\partial}_b \, K_\mu = -4\pi^2\mu. \tag{3}$$

Here $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator on ∂B; (3) means that

$$\int \phi \, d\mu = \frac{1}{4\pi^2}\int_{\partial B} K_\mu \, \overline{\partial}\phi \, \wedge \omega \tag{4}$$

for all functions ϕ smooth in a neighborhood of ∂B, where $\omega(z) = dz_1 \wedge dz_2$. An elementary proof of (4) is presented in H.P. Lee's thesis [14]; Varopoulos ([19], §3.2) has also given an exposition of Henkin's results for the case of the ball.

Note that the condition (2) that μ be orthogonal to polynomials (satified by all $\mu \in R(X)^\perp$) is necessary for the solution of (3), and that (3) implies that K_μ is a CR function on $\partial B \setminus X$. Lee and Wermer [15] proved that if X is rationally convex, then K_μ extends holomorphically from $\partial B \setminus X$ to B for any $\mu \in R(X)^\perp$:

Theorem 2.1 *Suppose X is a rationally convex subset of ∂B. Let μ be a measure on X such that $\mu \in R(X)^\perp$, and let K_μ be its Henkin transform. Then there exists a function k_μ, holomorphic in a neighborhood of $\overline{B} \setminus X$, with $k_\mu = K_\mu$ on $\partial B \setminus X$.*

We let $H^1(B)$ denote the Hardy space of functions g holomorphic on B satisfying

$$\sup\left\{\int_{\partial B} g^{(r)} \, d\sigma : r < 1\right\} < \infty$$

where $g^{(r)}(z) \equiv g(rz)$ for $z \in \partial B$. For $g \in H^1(B)$, $\lim_{r\to 1} g^{(r)} \equiv g^*$ exists σ - a.e on ∂B, and $g^{(r)} \to g^*$ as $r \to 1$ in $L^1(d\sigma)$.

Lemma 2.2 *Let X be a rationally convex subset of ∂B with $\sigma(X) = 0$. Let μ be a measure on X with $\mu \perp R(X)$, and let k_μ be the holomorphic extension of K_μ to B (as in Theorem 2.1). If $k_\mu \in H^1(B)$, then μ is the zero measure.*

Proof: It suffices to show that $\int \phi \, d\mu = 0$ for every function $\phi \in C^1(\mathbf{C}^2)$. Note that $\sigma(X) = 0$ implies that $k_\mu^* = K_\mu$ at σ-almost all points of ∂B, and so by (4)

$$\int_X \phi \, d\mu = \frac{1}{4\pi^2}\int_{\partial B} k_\mu^* \, \overline{\partial}\phi \, \wedge \omega = \lim_{r\to 1} \frac{1}{4\pi^2}\int_{\partial B} k_\mu^{(r)} \, \overline{\partial}\phi \, \wedge \omega$$

By Stokes' theorem, for fixed r

$$\int_{\partial B} k_\mu^{(r)} \, \overline{\partial}\phi \, \wedge \omega = \int_B \overline{\partial}(k_\mu^{(r)} \, \overline{\partial}\phi \wedge \omega) = \int_B \overline{\partial}(k_\mu^{(r)}) \wedge \overline{\partial}\phi \wedge \omega = 0$$

since $k_\mu^{(r)}$ is holomorphic in B. This shows that $\int \phi \, d\mu = 0$ for all $\phi \in C^1(\mathbf{C}^2)$ and completes the proof. $\square$

Thus to prove that $R(X) = C(X)$ for a rationally convex subset of ∂B with $\sigma(X) = 0$, it suffices to show that $k_\mu \in H^1(B)$ for every $\mu \perp R(X)$. We will use this approach to establish rational approximation on certain subsets of ∂B. It should be noted that the condition that $\sigma(X) = 0$ is necessary in the preceding lemma. If X is the rationally convex set constructed by Basener, $R(X) \neq C(X)$, and there exist nonzero measures $\mu \in R(X)^\perp$ for which $k_\mu \in H^1(B)$ ([4]).

We begin with a general estimate on the Henkin transform.

Lemma 2.3 *If $X \subset \partial B$, μ is a measure supported on X, and $z \in \partial B$, then*

$$|K_\mu(z)| \leq \frac{4\|\mu\|}{\mathrm{dist}^4(z, X)} \tag{5}$$

Proof: For any $\zeta, z \in \partial B$,

$$|z - \zeta|^2 = |z|^2 + |\zeta|^2 - 2 \,\mathrm{Re}(\langle z, \zeta\rangle) = 2 \,\mathrm{Re}(1 - \langle z, \zeta\rangle) \leq 2|1 - \langle z, \zeta\rangle|$$

and thus for $\zeta \in X, \ z \in \partial B$,

$$\mathrm{dist}^2(z, X) \leq 2|1 - \langle z, \zeta\rangle|. \tag{6}$$

We obtain from this an estimate on Henkin's kernel H: for $z \in \partial B, \zeta \in X$

$$|H(z, \zeta)| = \frac{|\langle Tz, \zeta\rangle|}{|1 - \langle z, \zeta\rangle|^2} \leq \frac{4|Tz||\zeta|}{\mathrm{dist}^4(z, X)} = \frac{4}{\mathrm{dist}^4(z, X)}$$

from which (5) follows immediately, by the definition of K_μ. $\square$

We would like to establish an estimate similar to (5) for the holomorphic extension k_μ of K_μ to B given by Theorem 2.1 for rationally convex X. We shall do this for the class of sets satisfying the following strong notion of convexity with respect to rational functions:

Definition: Given $X \subset \mathbf{C}^2$, let $X_\epsilon = \{z \in \mathbf{C}^n : \mathrm{dist}(z, X) < \epsilon\}$. We say that X has the *hull-neighborhood* property (abbreviated (H-N)) if there exists $k > 0$ such that, if we put $E = \pi(X)$, we have for all $\epsilon > 0$,

$$\widehat{[X_\epsilon]}_r \cap \pi^{-1}(E) \subset X_{k\epsilon}. \tag{7}$$

In other words, given $z \in \mathbf{C}^2$ with $\pi(z) \in \pi(X)$ and $\epsilon > 0$ so that $\operatorname{dist}(z, X) > k\epsilon$, there exists a polynomial Q with $Q(z) = 0$ whose zero set does not meet X_ϵ. Since $\pi(\widehat{X_r}) = \pi(X)$, it is clear that if X has property (H-N), then X is rationally convex. Also, for $X \subset \partial B$, $[X_\epsilon]\widehat{_r}$ is contained in the ball of radius $1+\epsilon$ centered at the origin, so $[X_\epsilon]\widehat{_r} \subset X_{2+\epsilon}$. Therefore for $X \subset \partial B$, there exists $k > 0$ such that (7) holds for all $\epsilon > 0$ if and only if there exists $k > 0$ such that (7) holds for all sufficiently small ϵ.

Lemma 2.4 *Assume $X \subset \partial B$ has property* (H-N). *Then there exists a constant c so that for all $p \in B$ with $\pi(p) \in \pi(X)$ and all $\mu \in R(X)^\perp$, we have*

$$|k_\mu(p)| \leq \frac{c||\mu||}{\operatorname{dist}^4(p, X)}. \tag{8}$$

Proof: Fix $p \in B$, set $\delta = \operatorname{dist}(p, X)$. If $\epsilon > 0$ satisfies $k\epsilon < \delta$, then by hypothesis $p \notin [X_\epsilon]\widehat{_r}$, so there exists a polynomial Q with $Q(p) = 0$ such that the zero set V of Q does not meet X_ϵ. Note that k_μ is continuous on $V \cap \overline{B}$ with boundary values K_μ on $V \cap \partial B$. By the maximum principle, $|k_\mu|$ attains its maximum on $V \cap \overline{B}$ at a point $p_0 \in \partial B \cap V$, and so by Lemma 2.3,

$$|k_\mu(p)| \leq |K_\mu(p_0)| \leq \frac{4||\mu||}{\operatorname{dist}^4(p_0, X)} \leq \frac{4||\mu||}{\epsilon^4}$$

Since the preceding inequality holds whenever $k\epsilon < \delta$, we obtain (8). $\square$

Let $\triangle$ denote the closed unit disk in the complex plane. For a function defined on $\triangle$, we let $\Gamma(f) \subset \mathbf{C}^2$ denote the graph of f over $\triangle$. $\operatorname{Lip}(\triangle)$ will denote the set of Lipschitz functions on $\triangle$, i.e, those functions f for which there exists a constant $M > 0$ such that $|f(z) - f(z')| \leq M|z - z'|$ for all $z, z' \in \triangle$; the least such M we call the Lipschitz constant for f. The main result of this section is the following approximation theorem for subsets of Lipschitz graphs with the hull-neighborhood property.

Theorem 2.5 *Let $f \in Lip(\triangle)$. Assume $\Gamma(f) \subset \partial B$. If $X \subset \Gamma(f)$ has property* (H-N), *then $R(X) = C(X)$.*

Proof: We will show that under the hypotheses of Theorem 2.5, $k_\mu \in H^1(B)$ for each $\mu \in R(X)^\perp$. By Lemma 2.2, since $\sigma(\Gamma(f)) = 0$ this will imply that every measure in $R(X)^\perp$ is identically zero, and hence $R(X) = C(X)$. Fix $\mu \in R(X)^\perp$, and write $k = k_\mu$. Let (z, w)

denote the coordinates in $\mathbf{C}^2$. We show that $k \in H^1(B)$ by estimating k on the slices $z =$ constant. To do this, we first introduce some notation and prove a lemma.

For $z \in \triangle$, let $D_z = \{w : |w| < \sqrt{1-|z|^2}\}$, and let γ_z be the boundary of D_z. If g is a function holomorphic in B and $z \in \triangle$, we let g_z denote the slice function $g_z(w) = g(z,w), w \in D_z$. If for some $s > 0$ we have $g_z \in H^s(D_z)$, i.e.,

$$\sup\{\int_0^{2\pi} |g_z(r\sqrt{1-|z|^2}e^{i\theta})|^s \, d\theta : 0 < r < 1\} < \infty \tag{9}$$

then $g_z^*(w) = \lim_{r\to 1} g_z(rw)$ exists for almost all $w \in \gamma_z$. If in addition $g_z^*(w) \in L^1$ with respect to linear measure on γ_z, then in fact $g_z \in H^1(D_z)$ (see [8], Theorem 2.11) and $\int_0^{2\pi} |g(z, r\sqrt{1-|z|^2}e^{i\theta})| \, d\theta$ is increasing in r.

Lemma 2.6 *Let X be a subset of ∂B with $\sigma(X) = 0$. Suppose g is holomorphic in a neighborhood of $\overline{B} \setminus X$, $g|_{\partial B} \in L^1(d\sigma)$, and for some $s > 0$, $g_z \in H^s(D_z)$ for almost all $z \in \triangle$. Then $g \in H^1(B)$.*

Proof: First note that if f is any positive function defined (σ - a.e.) on ∂B, then (see Proposition 1.47 of [17]),

$$\int_{\partial B} f \, d\sigma = \int_\triangle dm(z) \int_0^{2\pi} f_z(\sqrt{1-|z|^2}e^{i\phi}) \, d\phi \tag{10}$$

Set $G = g|_{\partial B}$. The hypotheses imply that for m-almost all $z \in \triangle$, we have $G|_{\gamma_z} = g_z^*$ is defined almost everywhere and integrable with respect to linear measure on γ_z, and $g_z \in H^1(D_z)$. Thus if $r < 1$, by (10)

$$\begin{aligned} \int_{\partial B} |g^{(r)}| \, d\sigma &= \int_\triangle dm(\zeta) \int_0^{2\pi} |g_{rz}(r\sqrt{1-|z|^2}e^{i\phi})| \, d\phi \\ &\le \int_\triangle dm(z) \int_0^{2\pi} |g_{rz}^*(\sqrt{1-|rz|^2}e^{i\phi})| \, d\phi \end{aligned}$$

The change of variables $z' = rz$ converts the last integral above to

$$\frac{1}{r^2}\int_{|z'|\le r} dm(z') \int_0^{2\pi} |G(z', \sqrt{1-|z'|^2}e^{i\phi})| \, d\phi \le \frac{1}{r^2}\int_{\partial B} |G| \, d\sigma$$

again by (10). Since $G \in L^1(d\sigma)$, we find that $\int_{\partial B} |g^{(r)}| \, d\sigma$ is bounded independently of r, so $g \in H^1(B)$. $\square$

By Lemma 2.6, the proof of Theorem 2.5 will be complete if we can show that for some $s > 0$, $k_z \in H^s(D_z)$ for almost all $z \in \triangle$. Fix $z \in \triangle$. We may assume $z \in \pi(X)$, for if $z \notin \pi(X)$, then k_z is holomorphic in a neighborhood of the closure of D_z, and there is nothing to prove. If $p = (z, w)$, with $w \in D_z$, then for any $p' = (z', f(z'))$,

$$\begin{aligned} |w - f(z)| &\leq |w - f(z')| + |f(z') - f(z)| \\ &\leq |w - f(z')| + M|z - z'| \\ &\leq \sqrt{M^2+1}\,|p - p'| \end{aligned}$$

by the Cauchy-Schwarz inequality, and so

$$|w - f(z)| \leq \sqrt{M^2+1}\ \mathrm{dist}(p, X) \tag{11}$$

By Lemma 2.4, then

$$|k(p)| \leq \frac{C}{\mathrm{dist}^4(p, X)} \leq \frac{C'}{|w - f(z)|^4} \tag{12}$$

for some constant C'. Write $f(z) = \sqrt{1-|z|^2}e^{i\phi}$. Then using (12), for $r < 1$ we obtain

$$\begin{aligned} \int_0^{2\pi} |k_z(r\sqrt{1-|z|^2}e^{i\theta})|^{1/8}\, d\theta &\leq \frac{C'}{(1-|z|^2)^{1/4}} \int_0^{2\pi} \frac{1}{|re^{i\theta} - e^{i\phi}|^{1/2}}\, d\theta \\ &= C'' \int_0^{2\pi} \frac{1}{|re^{i\theta} - 1|^{1/2}}\, d\theta \end{aligned}$$

For $|\theta| \leq \pi/3$, $\cos(\theta) \leq 1 - \theta^2/4$, which implies

$$|1 - re^{i\theta}|^{1/2} = [1 + r^2 - 2r\cos(\theta)]^{1/4} \geq [(1-r)^2 + \theta^2/4]^{1/4} \geq \sqrt{\theta}/\sqrt{2}$$

It follows from this that the last integral is bounded independently of r, and so $k \in H^{1/8}(D_z)$ for all $z \in \triangle$. This completes the proof. □

Remark: The special case of Theorem 2.5 when f is continuously differentiable on $\triangle$ can also be obtained as a direct consequence of Theorem 4.3 of ([2]).

We close this section by exhibiting a class of sets with the hull-neighborhood property. Recall that a real submanifold of $\mathbf{C}^n$ is said to be totally real if at each point, its tangent space contains no complex line.

Theorem 2.7 *Let $f \in C^\infty(\triangle)$, and assume $\Gamma(f)$ is a totally real submanifold of $\mathbf{C}^2$. If X is a compact polynomially convex subset of $\Gamma(f)$, then X has property* (H-N).

Proof: For $p \in \mathbf{C}^2$, let $\delta(p) = \mathrm{dist}(p, \Gamma(f))$. Since $\Gamma(f)$ is totally real, a result of Hörmander and Wermer ([12], or see [1], Lemma 17.2) implies that there is a neighborhood U of X in $\mathbf{C}^2$ such that δ^2 is strictly plurisubharmonic on U.

Since X is polynomially convex, there exists a compact polynomial polyhedron Π, $X \subset \Pi \subset U$, where $\Pi = \{|P_j| \leq 1, j = 1, \ldots, k\}$ with each P_j a polynomial. We may assume that $|P_j| \leq 1/2$ on X, for each j. Define a function Ψ on $\mathbf{C}^2$ by

$$\Psi = \max\{|P_1|, \ldots, |P_k|\} - \frac{3}{4}$$

Then $\Psi = 1/4$ on $\partial\Pi$ and $\Psi < 0$ on X.

Choose $\epsilon_0 > 0$ so small that $\Psi < 0$ on X_{ϵ_0}. We will show that whenever $p \in \mathbf{C}^2$ satisfies $\pi(p) \in \pi(X)$ and $\mathrm{dist}(p, X) > \sqrt{M^2+1}\,\epsilon$ for some $\epsilon < \epsilon_0$, where M is the Lipschitz constant for f, then there is a polynomial Q with $Q(p) = 0$ whose zero set does not meet X_ϵ. By the remarks following the definition of (H-N), this will complete the proof.

Choose a constant $\kappa > 0$ so that $\kappa\delta^2(p) < 1/4$ for all $p \in \partial\Pi$. Then on a neighborhood N of $\partial\Pi$ we have $\kappa\delta^2 < \Psi$. Define F as follows:

$$F = \begin{cases} \max(\Psi, \kappa\delta^2) & \text{on } \Pi \cup N \\ \Psi & \text{on } \mathbf{C}^2 \setminus \Pi \end{cases}$$

Then F is well-defined and plurisubharmonic on $\mathbf{C}^2$. For $\epsilon < \epsilon_0$ set

$$\Lambda_\epsilon = \{q \in \mathbf{C}^2 : F(q) \leq \kappa\epsilon^2\}$$

Then Λ_ϵ is compact, and $X_\epsilon \subset \Lambda_\epsilon$, for if $\mathrm{dist}(q, X) < \epsilon$, then $\Psi(q) < 0$, so

$$F(q) = \kappa\delta^2(q) \leq \kappa\ \mathrm{dist}^2(q, X) < \kappa\epsilon^2$$

implying $q \in \Lambda_\epsilon$. Also, since F is plurisubharmonic, Λ_ϵ is polynomially convex (this follows from [11], Theorem 4.3.4). Suppose p satisfies $\mathrm{dist}(p, X) > \sqrt{M^2+1}\,\epsilon$. We distinguish two cases: either (1) $F(p) = \kappa\delta^2(p)$, or (2) $F(p) = \Psi(p)$. In the first case, we find as in the proof of Theorem 2.5 that if we write p in coordinates as $p = (z, w)$ then $|w - f(z)| \leq \sqrt{M^2+1}\,|p - p'|$ whenever $p' \in \Gamma(f)$, implying $\mathrm{dist}(p, X) \leq \sqrt{M^2+1}\,\delta(p)$, and so

$$F(p) \geq \frac{\kappa\ \mathrm{dist}^2(p, X)}{M^2+1} > \kappa\epsilon^2$$

and thus $p \notin \Lambda_\epsilon$. By the polynomial convexity of Λ_ϵ, there exists a polynomial Q, nonvanishing on Λ_ϵ with $Q(p) = 0$; since $X_\epsilon \subset \Lambda_\epsilon$, Q does not vanish on X_ϵ. In the second case, we must have $\Psi(p) > 0$, and so $|P_j(p)| > 3/4$ for some j. Set $Q = P_j - P_j(p)$. Then $Q(p) = 0$, but since $\Psi < 0$ on X_ϵ, $|P_j| < 3/4$ on X_ϵ, so Q cannot vanish on X_ϵ. In both cases, we have found the required polynomial Q, and the proof is complete. $\square$

Finally we note that the approach in this section is related to the problem of determining when X is a removable singularity for integrable CR functions. In this context, we may say that X is removable for L^1 CR functions if X has the property that whenever $g \in L^1(d\sigma)$ and $\bar{\partial}_b g = 0$ off X, then g extends to a function in $H^1(B)$ (see [3]). By (3), $\bar{\partial}_b K_\mu = 0$ off X whenever $\mu \in R(X)^\perp$, and hence by the remarks following Lemma 2.2, $R(X) = C(X)$ for any subset of ∂B with $\sigma(X) = 0$ that is removable for L^1 CR functions. The paper [16] contains an extensive bibliography on this question and a survey of recent results.

3. The algebra generated by $R(E)$ and a smooth function

In this section we study the algebra generated by $R(E)$ and a smooth function on a planar set E. We then apply our results to the question of rational approximation on certain subsets of ∂B.

If $\mathcal{A}$ is a uniform algebra on a compact space X, we write $\mathcal{M}(\mathcal{A})$ for its maximal ideal space, and view elements of $\mathcal{M}(\mathcal{A})$ as homomorphisms $m : \mathcal{A} \to \mathbf{C}$. We will identify each point $x \in X$ with the point evaluation $m_x \in \mathcal{M}(\mathcal{A})$ defined by $m_x(h) = h(x)$. When $\mathcal{A} = R(X)$ for some compact subset $X \subset \mathbf{C}^n$, then $\mathcal{M}(\mathcal{A})$ can be identified with $\widehat{X}_r$ via $m \in \mathcal{M}(\mathcal{A}) \to (m(z_1), \ldots, m(z_n))$ where $(z_1, \ldots, z_n)$ are the coordinate functions. This correspondence is a homeomorphism.

If $\mathcal{F}$ is a family of continuous functions on a compact space X, then $[\mathcal{F}]$ will denote the algebra generated by $\mathcal{F}$, i.e., the smallest closed subalgebra of $C(X)$ containing $\mathcal{F}$. In [20], J. Wermer studied the algebra $\mathcal{A} = [z, f]$ on $\triangle$ generated by the identity function z and a smooth function f. Under the assumption that $\mathcal{M}(\mathcal{A}) = \triangle$, he showed that $\mathcal{A}$ consists of those continuous functions on $\triangle$ whose restrictions to the zero set E of $\partial f/\partial \bar{z}$ lie in $R(E)$. We will make use of the following generalization of Wermer's result due to Anderson and

Izzo ([2], Theorem 4.2):

Lemma 3.1 *Let $\mathcal{G}$ be a collection of continuously differentiable functions on $\triangle$, and set $\mathcal{A} = [\mathcal{G}]$. Assume the function z lies in $\mathcal{A}$, and that $\mathcal{M}(\mathcal{A}) = \triangle$. Set $T = \{\zeta \in \triangle : \frac{\partial g}{\partial \bar{z}}(\zeta) = 0,\ \forall g \in \mathcal{G}\}$. Then $\mathcal{A} = \{g \in C(\triangle) : g|_T \in R(T)\}$.*

In order to pass from algebras on compact subsets of the disk to algebras on the disk, we will need two results on extension algebras. The first is due to Bear [6] :

Lemma 3.2 *Let $\mathcal{A}_0$ be a uniform algebra on a compact subset X_0 of a compact space X. Put $\mathcal{A} = \{h \in C(X) \ : \ h|_{X_0} \in \mathcal{A}_0\}$. If $\mathcal{M}(\mathcal{A}_0) = X_0$, then $\mathcal{M}(\mathcal{A}) = X$.*

Lemma 3.3 *Let $\mathcal{A}$, $\mathcal{A}_0$, X, and X_0 be as in Lemma 3.2. Assume $\mathcal{G}_0$ is a subset of $C(X_0)$ with $[\mathcal{G}_0] = \mathcal{A}_0$. Let $\mathcal{G} \subset C(X)$ and assume (1) $[\mathcal{G}]$ contains all continuous functions on X vanishing in a neighborhood of X_0, and (2) $\mathcal{G}|_{X_0} = \mathcal{G}_0$. Then $[\mathcal{G}] = \mathcal{A}$.*

Proof: Clearly $\mathcal{G} \subset \mathcal{A}$, and so it suffices to show, given $h \in \mathcal{A}$, that $\int h \, d\mu = 0$ for all measures $\mu \in [\mathcal{G}]^{\perp}$. For any such measure the hypothesis that $[\mathcal{G}]$ contains all continuous functions vanishing near X_0 implies $\mathrm{supp}(\mu) \subset X_0$. Since $h|_{X_0} \in \mathcal{A}_0$, we may choose a sequence h_j of polynomials in elements of $\mathcal{G}_0$ converging to h on X_0. By hypothesis (2), we may assume each h_j is the restriction to X_0 of an element of $[\mathcal{G}]$. Then

$$\int_X h \, d\mu = \int_{X_0} h \, d\mu = \lim_{j \to \infty} \int_{X_0} h_j \, d\mu = 0$$

since $\mu \in [\mathcal{G}]^{\perp}$. $\Box$

Given a compact $E \subset \mathbf{C}$, we write $f \in C^1(E)$ if f is the restriction to E of a function continuously differentiable in some neighborhood of E.

Theorem 3.4 *Let E be a compact subset of $\mathbf{C}$, and take $f \in C^1(E)$. Assume $\mathcal{M}([R(E), f]) = E$. If $[R(E), f] \neq C(E)$, then there exists a compact subset E_0 of E such that $R(E_0) \neq C(E_0)$ and $f|_{E_0} \in R(E_0)$.*

Proof: Let E and f satisfy the hypotheses of the theorem. Without loss of generality, E is a compact subset of the open unit disk. Set $\mathcal{A} = \{h \in C(\triangle) \ : \ h|_E \in [R(E), f]\}$. Since $\mathcal{M}([R(E), f]) = E$ by hypothesis, Lemma 3.2 implies that $\mathcal{M}(\mathcal{A}) = \triangle$. Fix any smooth extension of f to $\triangle$ (we denote the extension by f, also). Since $R(E)$ is generated by the set of functions holomorphic in a neighborhood of E, Lemma 3.3 implies that $\mathcal{A}$ is generated by the set $\mathcal{G}$ consisting of f together with all functions smooth on $\triangle$ and holomorphic in a neighborhood of E. Set $E_0 = \{\zeta \in \triangle : \partial g/\partial\bar{z}(\zeta) = 0,\ \forall g \in \mathcal{G}\}$. Clearly $E_0 \subset E$. By Lemma 3.1, $\mathcal{A} = \{h \in C(X) : h|_{E_0} \in R(E_0)\}$. Since $f \in \mathcal{A}$, $f|_{E_0} \in R(E_0)$. If $R(E_0) = C(E_0)$, then $\mathcal{A} = C(X)$ and hence $[R(E), f] = C(E)$, contrary to hypothesis. $\Box$

As mentioned in the introduction, Basener gave examples of rationally convex subsets X of ∂B with $R(X) \neq C(X)$. To explain Basener's construction, we recall the notion of a Jensen measure. Given a uniform algebra $\mathcal{A}$ on X, a probability measure σ on X is said to be a *Jensen measure for* $m \in \mathcal{M}(\mathcal{A})$ if for every $h \in \mathcal{A}$,

$$\log |m(h)| \leq \int_X \log |h|\, d\sigma.$$

If m is point evaluation at some $p_0 \in X$, the point mass δ_{p_0} at p_0 is trivially a Jensen measure for m. Every Jensen measure σ for m represents m: $m(h) = \int h\, d\sigma$ for all $h \in \mathcal{A}$. Basener's assumption for $X \subset \partial B$ was the following condition on $E = \pi(X)$:

(B) For all $z_0 \in E$ the only Jensen measure for z_0 relative to $R(E)$ is δ_{z_0}.

It can be shown (see [7], Theorem 3.4.11) that (B) is equivalent to the condition that the set of functions harmonic in a neighborhood of E is dense in $C(E)$. Examples of sets $E \subset \mathbf{C}$ satisfying (B) for which $R(E) \neq C(E)$ can be found in [7], p. 193 ff. and [18], §27.

Basener showed that if $X \subset \partial B$ has the form $X = \{(z, w) \in \partial B : z \in E\}$ where E is a compact subset of the open unit disk satisfying (B), then X is rationally convex; in fact, his proof shows (see also [18], §19.8) that the same is true for any $X \subset \partial B$ for which $\pi(X) = E \subset \mathrm{int}(\triangle)$ satisfies (B). Our next lemma has a similar flavor:

Lemma 3.5 *Let E be a compact subset of* $\mathbf{C}$ *satisfying* (B), *and let $f \in C(E)$. Then* $\mathcal{M}([R(E), f]) = E$.

This can be proved by an argument essentially the same as that of Basener mentioned above, but a simpler approach is to note that it is an immediate consequence of the following easy lemma (which strengthens Lemma 2.2 of [13]).

Lemma 3.6 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras on a compact space X and $\mathcal{A} \subset \mathcal{B}$. If $x \in X$ is such that the only Jensen measure for x relative to $\mathcal{A}$ is δ_x, and $m \in \mathcal{M}(\mathcal{B})$ coincides with point evaluation at x when restricted to $\mathcal{A}$, then m is point evaluation at x on all of $\mathcal{B}$.*

Proof: Let μ be a Jensen measure for m (as a functional on $\mathcal{B}$). Then obviously μ is a Jensen measure for the restriction of m to $\mathcal{A}$, i.e., for point evaluation at x on $\mathcal{A}$. Hence by hypothesis $\mu = \delta_x$. Since μ represents m, we conclude that m is point evaluation at x on all of $\mathcal{B}$. $\Box$

If $\mathcal{A}$ is a uniform algebra on X, a point $p \in X$ is a peak point for A if there exists a function $f \in A$ with $f(p) = 1$ while $|f| < 1$ on $X \setminus \{p\}$. When X is a compact planar set, Bishop proved that $R(X) = C(X)$ if almost every point of X is a peak point for $R(X)$.

Theorem 3.7 *Let E be a compact subset of $\mathbf{C}$ satisfying* (B), *and let $f \in C^1(E)$. If almost every point of E is a peak point for $[R(E), f]$, then $[R(E), f] = C(E)$.*

Proof: Suppose that $[R(E), f] \neq C(E)$. By Lemma 3.5, $\mathcal{M}([R(E), f]) = E$. We may then apply Theorem 3.4 to produce a compact subset E_0 of E with $f|_{E_0} \in R(E_0)$ and $R(E_0) \neq C(E_0)$. If $z \in E_0$ is a peak point for $[R(E), f]$, choose $g \in [R(E), f]$ peaking at z. Since $g|_{E_0} \in R(E_0)$, the point z is a peak point for $R(E_0)$. By Bishop's peak-point theorem, $R(E_0) = C(E_0)$, which is a contradiction. $\Box$

Corollary 3.8 *Let E be a compact subset of the open unit disk satisfying* (B), *let $f \in C^1(E)$, and set $X = \{(z, f(z)) : z \in E\}$. If $X \subset \partial B$, then $R(X) = C(X)$.*

Proof: Let $\mathcal{A}$ be the algebra on X generated by $r(z)$ and w, where (z, w) are coordinates in $\mathbf{C}^2$ and r ranges over $R(E)$. Since $\mathcal{A} \subset R(X)$, it suffices to show that $\mathcal{A} = C(X)$. Moreover, $\mathcal{A}$ is isometrically isomorphic to the algebra on E generated by $R(E)$ and f, and therefore it is enough to show $[R(E), f] = C(E)$. Each point of ∂B is a peak point for polynomials, hence is a peak point for $\mathcal{A}$, and so every point of E is a peak point for $[R(E), f]$. By Theorem 3.7, $[R(E), f] = C(E)$. $\Box$

It is reasonable to conjecture that Theorems 3.4 and 3.7 remain valid if the hypothesis that $f \in C^1(E)$ is replaced by the assumption that f is merely continuous on E. We have no proof or counterexample.

Finally, we remark that Theorem 3.7 can also be obtained in a different fashion by combining our Lemma 3.5 with Theorem 4.3 of [2].

4. Approximation on Hölder graphs

In this section we show that the hypothesis $f \in \mathrm{Lip}(\triangle)$ of Theorem 2.5 may be weakened to the assumption that f satisfies a Hölder condition with exponent α, $0 < \alpha < 1$, on $E = \pi(X)$. That is, we assume there exists M so that for all $z, z' \in E$,

$$|f(z) - f(z')| \leq M|z - z'|^\alpha \tag{13}$$

To establish Theorem 2.5 under the hypothesis that f satisfies (13), it suffices to show (cf. (11) in the proof of Theorem 2.5) that there exists a constant C so that for $z \in E$, $w \in D_z$,

$$|w - f(z)| \leq C \operatorname{dist}((z, w), X)^\alpha \tag{14}$$

From (14) it follows, as in the proof of Theorem 2.5, that if $p = (z, w)$, we have the estimate

$$|k(p)| \leq \frac{C'}{|w - f(z)|^{4/\alpha}}$$

from which we infer $k \in H^{\alpha/8}(D_z)$ for all $z \in \triangle$, completing the proof.

To establish (14), we fix $p = (z, w)$, and take $p' = (z', f(z')) \in X$ so that $\operatorname{dist}(p, X) = |p - p'|$. Then

$$\begin{aligned} |w - f(z)| &\leq |w - f(z')| + |f(z') - f(z)| \\ &\leq |w - f(z')| + M|z - z'|^\alpha \\ &\leq (M^2 + 1)^{1/2}(|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/2} \end{aligned}$$

and so

$$\frac{|w - f(z)|^{2/\alpha}}{\operatorname{dist}^2(p, X)} \leq \frac{(M^2 + 1)^{1/\alpha}(|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/\alpha}}{|w - f(z')|^2 + |z - z'|^2} \tag{15}$$

Set $x = |w - f(z')|$, $y = |z - z'|$. Note $\operatorname{dist}^2(p, X) = x^2 + y^2 \leq 4$, since p, p' are points in the closed unit ball. The quantity

$$G(x, y) = \frac{(x^2 + y^{2\alpha})^{1/\alpha}}{x^2 + y^2}$$

on the right of (15) is clearly bounded on $1 \le x^2+y^2 \le 4$, so to complete the proof of (14), it suffices to show that $G(x,y)$ is bounded for $x^2+y^2<1$. Applying the elementary inequality $(A+B)^p \le 2^p(A^p+B^p)$ for positive A, B, p, we obtain

$$(x^2+y^{2\alpha})^{1/\alpha} \le 2^{1/\alpha}(x^{2/\alpha}+y^2) \le 2^{1/\alpha}(x^2+y^2)$$

using, in the last inequality, the fact that $x<1$. Therefore, $G(x,y) \le 2^{1/\alpha}$ for $x^2+y^2<1$, and the proof is finished.

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