College of the Holy Cross
Math 136, Fall Semester, 2005
Exercises on Sequences and Series
Professor Hwang

1. A sequence (a_k) is defined recursively by

$$a_{k+1} = 2a_k - 1, \qquad a_0 = 2.$$

Calculate the terms up to and including a_6.

Solution $a_1 = 3$, $a_2 = 5$, $a_3 = 9$, $a_4 = 17$, $a_5 = 33$, $a_6 = 65$.

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2. A sequence (a_k) is defined recursively by

$$a_{k+1} = \frac{1}{2}\Big(a_k + \frac{3}{a_k}\Big), \qquad a_0 = 2.$$

Calculate the terms up to and including a_3 as fractions. According to your calculator, what is $|\sqrt{3} - a_3|$?

Solution Repeated calculation gives

$$a_1 = \tfrac{1}{2}(2 + \tfrac{3}{2}) = \tfrac{7}{4}, \qquad a_2 = \tfrac{1}{2}(\tfrac{7}{4} + \tfrac{12}{7}) = \tfrac{97}{56}, \qquad a_3 = \tfrac{1}{2}(\tfrac{97}{56} + \tfrac{168}{97}) = \tfrac{18817}{10864}.$$

Thus $|\sqrt{3} - a_3| \approx 0.000\,000\,002 = 2 \times 10^{-9}$

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3. For each series, write out the first five terms, and test for convergence. If possible, use more than one convergence test and confirm that you reach the same conclusion.

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1}, \qquad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}, \qquad \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}, \qquad \sum_{k=1}^{\infty} \frac{1}{e^k\sqrt{k}}, \qquad \sum_{k=0}^{\infty} \frac{(-1)^k \cdot k}{2k+1}.$$

Solution

$$\sum_{k=0}^{\infty} \frac{1}{k^2+1} = 1 + \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots$$

converges by the integral test, or by comparison with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{17}} + \cdots$$

diverges by the integral test, limit comparison with the harmonic series, or comparison with half the harmonic series (a sneaky trick; please see me for details).

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} + \cdots$$

converges by the alternating series test.

$$\sum_{k=1}^{\infty} \frac{1}{e^k\sqrt{k}} = \frac{1}{e} + \frac{1}{e^2\sqrt{2}} + \frac{1}{e^3\sqrt{3}} + \frac{1}{e^4\sqrt{4}} + \frac{1}{e^5\sqrt{5}} + \cdots$$

converges by the ratio test, or by comparison with $ds \sum_{k=1}^{\infty} \frac{1}{e^k}$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k \cdot k}{2k+1} = -\frac{1}{3} + \frac{2}{5} - \frac{3}{7} + \frac{4}{9} - \frac{5}{11} + \cdots$$

diverges by the vanishing criterion: the terms do not approach zero.

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4. Use the ratio test on the following series. If the ratio test is inconclusive, find a definitive test.

$$\sum_{k=1}^{\infty} \frac{1}{k}, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Solution The ratio test is inconclusive for both. Instead, use the integral test, or note that we analyzed both series in class.

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5. Use the ratio test to determine the x intervals on which the following series converge.

$$\sum_{k=0}^{\infty} (k!)x^k, \qquad \sum_{k=1}^{\infty} \frac{x^k}{k \cdot 2^k}.$$

Solution The first was done in class; the series converges only if $x = 0$. For the second, use the ratio test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{x^{k+1}}{(k+1)\cdot 2^{k+1}} \cdot \frac{k \cdot 2^k}{x^k}\right| = \lim_{k\to\infty} \left|\frac{x}{2} \cdot \frac{k}{k+1}\right| = \frac{|x|}{2}.$$

If this limit is smaller than 1, the series converges, while if the limit is larger than 1, the series diverges. Consequently, the series converges if $|x| < 2$, or $-2 < x < 2$, and diverges if $|x| > 2$. To check the endpoints $x = \pm 2$, plug them into the original series: At $x = 2$, we have the harmonic series, since

$$\sum_{k=1}^{\infty} \frac{2^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Thus, our series diverges at $x = 2$. Similarly, at $x = -2$ the series becomes the alternating harmonic series, which converges by the alternating series test. To summarize the second power series has radius 2, and the interval of convergence is $[-2, 2)$.