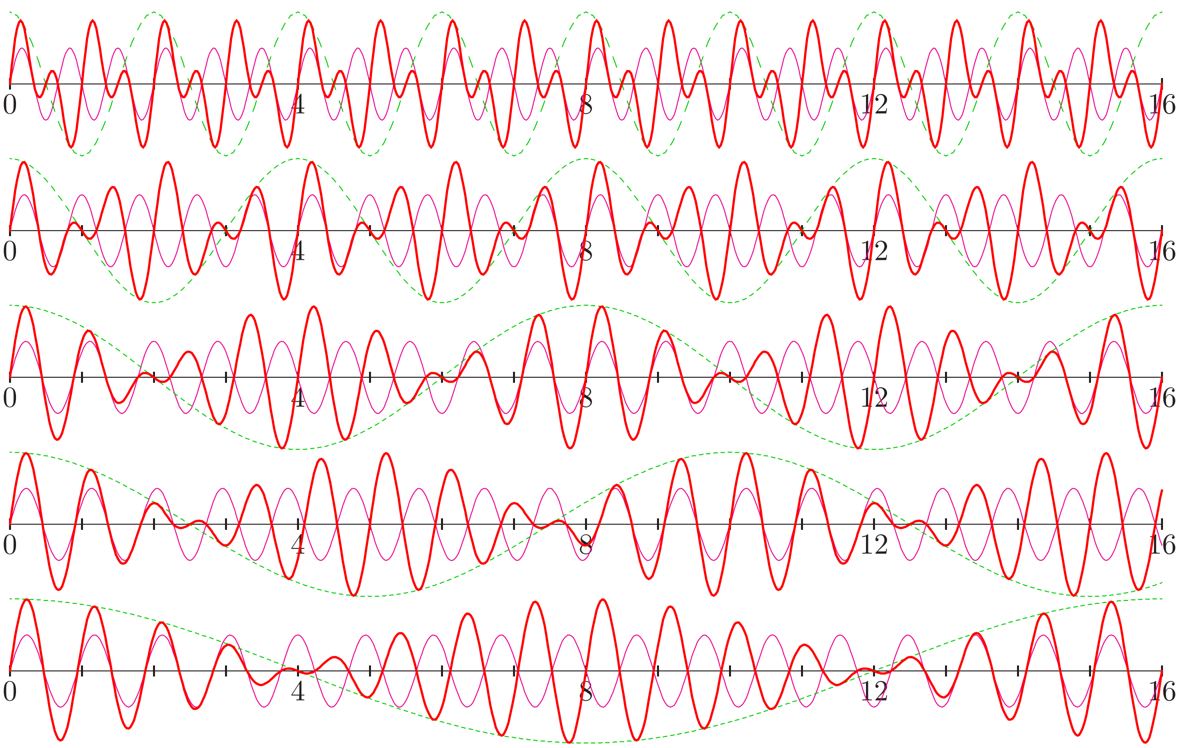
College of the Holy Cross, Fall Semester, 2005
Math 136, Lab #2 Solutions (Professor Hwang)
Trigonometry, Signals, and Music

Lab Questions

Question 1. `M := 1; N := 2; f := x -> S(M*x) + S(N*x);` Plot over $[0, 16]$. Keeping M fixed, successively re-plot for $N = 1.5$, 1.25, 1.2, and 1.125. In each case, estimate the frequency of the carrier and envelope; tabulate your findings, and determine (by guessing) formulas in terms of M and N.

Solution The graphs are shown below in red, with the (calculated!) carrier in pink and the envelope in green:



The beat wavelength, a.k.a. the reciprocal of the beat frequency, seems to be respectively 2, 4, 8, 10, and 16. Inspired guessing might reveal the frequency to be half the difference of M and N. In each case, the carrier frequency (best observed in the last three graphs) seems to be about 1.

Question 2. Use the trig identities

$$\sin(u \pm v) = (\sin u)(\cos v) \pm (\sin v)(\cos u).$$

to express $2 \sin(u) \cos(v)$ as a sum of two sine functions of different frequencies.

Solution Adding the identities

$$\sin(u+v) = (\sin u)(\cos v) + (\sin v)(\cos u),$$
$$\sin(u-v) = (\sin u)(\cos v) - (\sin v)(\cos u)$$

gives $\sin(u+v) + \sin(u-v) = 2(\sin u)(\cos v)$.

Question 3. If M and N are positive real numbers, find a and b so that

$$\sin(Mx) + \sin(Nx) = 2\sin(ax)\cos(bx).$$

Solution Comparing the desired identity with the result of Question 2, we are led to the equations $u+v = Mx$ and $u-v = Nx$. Adding the equations gives $2u = (M+N)x$, or $u = \frac{1}{2}(M+N)x$. Similarly, subtracting gives $v = \frac{1}{2}(M-N)x$. The result of Question 2 says that

$$\sin(Mx) + \sin(Mx) = \sin(u+v) + \sin(u-v) = 2(\sin u)(\cos v) = 2(\sin \tfrac{M+N}{2}x)(\cos \tfrac{M-N}{2}x).$$

The right-hand side has the desired form if $a = \frac{1}{2}(M+N)$ and $b = \frac{1}{2}(M-N)$.

Question 4. Find the carrier and beat frequencies of $\sin(Mx) + \sin(Nx)$ as functions of M and N. Are your results consistent with your data from Question 1? For fixed N, what happens to the beat frequency as $M \to N$?

Solution The larger frequency, $a = \frac{1}{2}(M+N)$, must be the carrier and the smaller, $b = \frac{1}{2}(M-N)$, is the beat frequency. The beat frequency agrees completely with the last three graphs in Question 1, and is believable for the first two. The predicted carriers (1.5, 1.25, 1.125, 1.1, and 1.0625) are so close to 1 that they're difficult to distinguish visually from 1. Thus these formulas do seem to explain the data collected earlier. As $M \to N$, the beat frequency goes to zero. If you've tuned a guitar, you know that when two strings are slightly out of tune they "warble" slightly when plucked simultaneously. The warbling results from the sinusoidal variation in amplitude of the combined wave. The beats get slower as the strings are brought into tune, and disappear when the strings are exactly in tune.

Question 5. The human ear tends to prefer chords whose constituent notes have relative frequencies forming a ratio of small whole numbers, such as 2-to-1 (an octave), 3-to-2 (a fifth interval), or 4-to-3 (a fourth interval. If notes of frequency 1056 Hz and 352 Hz are played together, what are the carrier and beat frequencies? Will the result sound melodious or harsh? Explain.

Solution We have $M = 1056$ and $N = 352$. When combined, we'll hear a beat frequency of $\frac{1}{2}(1056 - 352) = 352$ and a carrier of $\frac{1}{2}(1056 + 352) = 704$. The beat frequency is exactly N, the ratio of carrier to beat is one octave, and the ratio of M to the carrier is $1056/704 = 3/2$, a fifth interval. These notes are likely to sound melodious together.