College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 10
Surfaces

Exercise 10.1: Assume $0 < r < R$ are real. Determine where the following mappings are regular surfaces:

(a) $\mathbf{x}(u, v) = R(\cos u \cos v, \sin u \cos v, \sin v)$.

(b) $\mathbf{x}(u, v) = \big((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v\big)$.

(c) $\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, \sinh u)$.

(d) $\mathbf{x}(u, v) = (\sinh u \cos v, \sinh u \sin v, v)$.

(e) $\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$.

Answer For each, calculate the partials and check for the cross product being nowhere zero. The first is non-regular where $\cos v = 0$ and regular everywhere else. The other four are regular everywhere.

Exercise 10.2: Suppose α and β are regular paths and $\|\beta\| = 1$. Determine conditions under which the mapping $\mathbf{x}(u, v) = \alpha(u) + v\beta(u)$ is a regular surface.

Answer The partials are $\mathbf{x}_u = \alpha'(u) + v\beta'(u)$ and $\mathbf{x}_v = \beta(u)$. This parametrization is regular provided the cross product is non-zero, i.e., when the partials are not proportional. General necessary *and* sufficient conditions boil down to this, but we can give necessary *or* sufficient conditions: If $\alpha'(u)$ and $\beta'(u)$ are proportional, there is a real v making $\mathbf{x}_u = \mathbf{0}$; otherwise $\mathbf{x}_u$ is non-zero for all real v. If $\beta(u)$ does not lie in the plane spanned by $\alpha'(u)$ and $\beta'(u)$, then the cross product is non-zero for all real v.

Exercise 10.3: Suppose $\alpha(u) = \big(x(u), 0, z(u)\big)$ is a regular path with $x > 0$. Find a parametrization $\mathbf{x}(u, v)$ of the surface obtained by revolving α about the z-axis, with v equal to the "longitude" on the surface, and show your parametrization is regular.

Answer $\mathbf{x}(u, v) = \big(x(u)\cos v, x(u)\sin v, z(u)\big)$. Regularity is straightforward to check.

Exercise 10.4: Let α be a regular path, and let $r > 0$. Find a parametrization of the "tube of radius r about α", i.e., the union of the circles of radius r in the normal planes. Suggestion: Express your parametrization in terms of the Frenet frame of α.

Answer One possibility is $\mathbf{x}(u, v) = \alpha(u) + r\cos v\mathbf{N} + r\sin v\mathbf{B}$, with the normal and binormal vectors evaluated at $\alpha(u)$.

Exercise 10.5: Show the plane $x = 1$ cuts the hyperboloid of one sheet, $x^2 + y^2 - z^2 = 1$, in a pair of lines crossing at right angles.

Answer Solving the given equations simultaneously gives $y^2 - z^2 = 0$, which is the equation of two lines $z = \pm y$ in the plane $x = 1$) meeting at right angles.