College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 09
Spherical Triangles, Geography

Exercise 09.1: A *circle* on a sphere S is a curve obtained by cutting S with a plane. If $R > 0$ is a real number and S is the sphere of radius R centered at the origin, show that every latitude line and every longitude line is a circle, and give parametric descriptions of each.

Answer A latitude on S is the intersection with a plane $z = z_0$ with $-R \leq z_0 \leq R$. A longitude is the intersection with a plane containing the z-axis. Parametric descriptions may be read off the geographic parametrization of the sphere,

$$\mathbf{x}(u, v) = (R\cos u \cos v, R\sin u \cos v, R\sin v),$$

since longitudes and latitudes are parameter curves. The curve at longitude θ_0 or at latitude φ_0 may be parametrized by

$$\begin{aligned}\operatorname{long}_{\theta_0}(t) &= (R\cos\theta_0 \cos t, R\sin\theta_0 \cos t, R\sin t),\\ \operatorname{lat}_{\varphi_0}(t) &= (R\cos t \cos\varphi_0, R\sin t \cos\varphi_0, R\sin\varphi_0).\end{aligned}$$

Exercise 09.2: A *great circle* on a sphere is a circle whose radius is equal to the radius of the sphere. Show that if p and q are points on a sphere, there exists at least one great circle containing both points. Under what conditions is the great circle unique? Hint for the second: Every plane determing a great circle passes through the center of the sphere.

Answer Let S be a sphere with center O. The points O, p, and q lie in at least one plane, and intersecting this plane with S yields a great circle containing both p and q. This great circle is unique provided O, p, and q are not collinear, i.e., p and q are distinct and not antipodal.

Exercise 09.3: A *semi-lune* on a sphere is a region bounded by two half-arcs of great circles, such as the region of a globe bounded by two portions of longitudes from the south pole to the north. If the sphere has radius R and the interior angle is θ, find the area of the resulting semi-lune.

Answer The area of the entire sphere is $4\pi R^2$, and a semi-lune of interior angle θ comprises $\theta/(2\pi)$ of the total area, or $2\theta R^2$.

Exercise 09.4: Assume the earth is a perfect sphere of radius 6370 km, which turns once on its axis every 24 hours and whose axis is tilted 23° from the orbital axis. If we stand at latitude φ at sunset on an equinox, how rapidly does the shadow rise from the surface? Particularly, how high is the shadow after one minute? Two minutes? Five minutes? Ten minutes? Express your answer in suitable units.

Answer See `https://math.stackexchange.com/questions/4181269`.