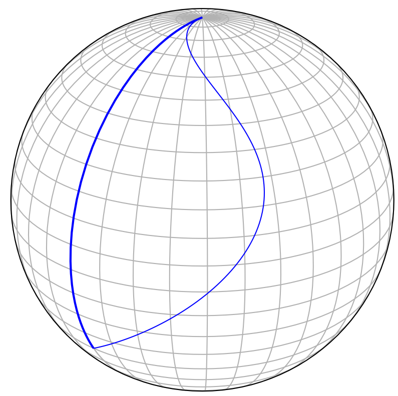
College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 07
The Fundamental Theorem of Curves

Exercise 07.1: Exercise 1.2.3(a) and (c), page 18.

Answer See answers in the back.

Exercise 07.4: Exercise 1.3.1, page 31. (a) Prove that the shortest path between two points on the unit sphere is an arc of a great circle connecting them. (b) Prove that if P and Q are points on the unit sphere, then the shortest path between them has length $\arccos(P \cdot Q)$.



Solution (a) As the book notes, we may choose coordinates so that $P = (0, 0, 1)$ and $Q = (\sin u_0, 0, \cos u_0)$ for some real u_0 in $(0, \pi]$. Every path from P to Q may be parametrized by picking functions u and v with $\big(u(a), v(a)\big) = (0, 0)$ and $\big(u(b), v(b)\big) = (\sin u_0, \cos u_0)$ and interpreting u as colatitude (angle measured from the north pole) and v as longitude on the unit sphere:

$$\alpha(t) = \big(\sin u(t) \cos v(t), \sin u(t) \sin v(t), \cos u(t)\big)$$

for $a \leq t \leq b$.

Intuitively, decreasing u moves us away from Q, and varying v diverts us to the side, so to minimize the arc length of α the function u should increase monotonically and v should be identically 0. We'll check this with calculus.

The velocity of α is

$$\alpha'(t) = \begin{bmatrix} \cos u(t) \cos v(t) \\ \cos u(t) \sin v(t) \\ -\sin u(t) \end{bmatrix} u'(t) + \begin{bmatrix} -\sin u(t) \sin v(t) \\ \sin u(t) \cos v(t) \\ 0 \end{bmatrix} v'(t).$$

As with paths in polar coordinates, the vector components are orthonormal. Consequently, the speed is $|\alpha'(t)| = \sqrt{u'(t)^2 + v'(t)^2} \geq |u'(t)|$, with equality if and only if $v'(t) \equiv 0$. Further, $|u'(t)| \geq u'(t)$ with equality if and only if $u'(t) \geq 0$, i.e., u is non-decreasing. We conclude that the arc length of α satisfies

$$\int_a^b |\alpha'(t)|\, dt \geq \int_a^b u'(t)\, dt = u(b) - u(a) = u_0,$$

with equality if and only if $v(t) = 0$ and $u'(t) \geq 0$ for all t with $a \leq t \leq b$.

(b) Geometry shows the longitude from P to Q has length $u_0 = \arccos(P \cdot Q)$.

Exercise 07.5: Exercise 1.3.5, page 31.

Idea The terms comprising f are of the form $\widetilde{\mathbf{T}} \cdot \widetilde{\mathbf{T}} - 2\widetilde{\mathbf{T}} \cdot \mathbf{T}^* + \mathbf{T}^* \cdot \mathbf{T}^* = 2 - 2\widetilde{\mathbf{T}} \cdot \mathbf{T}^*$, so the function f here is 6 minus twice the function in the proof of Theorem 3.1.