College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 05
The Frenet Equations

Exercise 05.1: (Linear algebra warm-up) Suppose $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is an ordered orthonormal triple in $\mathbf{R}^3$.

(a) Prove the set $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is linearly independent. That is, if $a\mathbf{T} + b\mathbf{N} + c\mathbf{B} = \mathbf{0}$ for some real a, b, c, then $a = b = c = 0$.

Answer Dotting the given condition with $\mathbf{T}$ shows $a = 0$. Similarly, dotting with $\mathbf{N}$ shows $b = 0$, and dotting with $\mathbf{B}$ shows $c = 0$.

(b) Prove the set $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ spans $\mathbf{R}^3$. That is, every vector v can be written as a linear combination. Hint: If $v = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$, dot each side with the orthonormal elements in turn to express a, b, and c in terms of v. Then show that the resulting components do in fact express v as a linear combination.

Answer If $a = v \cdot \mathbf{T}$, $b = v \cdot \mathbf{N}$, and $c = v \cdot \mathbf{B}$, part (a) shows $v - (a\mathbf{T} + b\mathbf{N} + c\mathbf{B}) = \mathbf{0}$.

(c) Let I be an open interval of real numbers, and let $f : I \to \mathbf{R}^3$ be a smooth mapping. Prove that the image of f lies on a sphere centered at the origin if and only if $f \cdot f' = 0$.

Answer Since $f \cdot f' = (\frac{1}{2}\|f\|^2)'$, the left-hand side is identically zero if and only if $\|f\|^2$ is constant, i.e., f lies on a sphere centered at the origin.

Exercise 05.2: Suppose α is a regular path in $\mathbf{R}^3$.

(a) Prove that the curvature κ is given by $\kappa = \dfrac{|\alpha' \times \alpha''|}{|\alpha'|^3}$. Hints: If v denotes the speed, then $\alpha' = v\mathbf{T}$ and $\alpha'' = v'\mathbf{T} + \kappa v^2\mathbf{N}$.

Answer Proposition 2.2., page 14.

(b) Prove that the torsion τ is given by $\tau = \dfrac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2}$.

Solution Since $\alpha'(t) = v\mathbf{T}$ is a scalar multiple of $\mathbf{T}$, it suffices to calculate the $\mathbf{T}$ component of $\alpha'' \times \alpha'''$. The only way to get $\mathbf{T}$ as a cross product from the Frenet frame is $\mathbf{N} \times \mathbf{B}$. Since α'' has no $\mathbf{B}$ component, we only need to calculate the $\mathbf{N}$ component of α'' and the $\mathbf{B}$ component of α'''.

Let s denote the arc length function of α, and as on page 13 use Leibniz notation to write the chain rule in the form $\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v\,\frac{d}{ds}$. The Frenet equations for an arclength-parametrized curve give

$$\frac{d\mathbf{N}}{dt} = v\,\frac{d\mathbf{N}}{ds} = v\,\mathbf{N}' = v(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}).$$

Since $\alpha'' = v'\mathbf{T} + \kappa v^2\mathbf{N}$ (the formula just above Proposition 2.2), we find

$$\alpha'''(t) = \cdots + \kappa v^2 \frac{d\mathbf{N}}{dt} = \cdots + \kappa v^3 \tau\, \mathbf{B}.$$

The $\mathbf{T}$ component of $\alpha'' \times \alpha'''$ is $(\kappa v^2\mathbf{N}) \times (\kappa v^3\tau\mathbf{B}) = \kappa^2 v^5 \tau \mathbf{T}$, so

$$\frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2} = \frac{\kappa^2 v^6 \tau}{(\kappa v^3)^2} = \tau.$$

(c) Calculate the curvature and torsion of the twisted cubic $\alpha(t) = (at, bt^2, ct^3)$.

Answer We have $\alpha'(t) = (a, 2bt, 3ct^2)$, $\alpha''(t) = (0, 2b, 6ct)$, $\alpha''(t) = (0, 0, 6c)$. Thus $v = (a^2 + 4b^2t^2 + 9c^2t^4)^{1/2}$, $\alpha' \times \alpha'' = (6bct^2, -6act, 2ab)$, and $\alpha'' \times \alpha''' = (12bc, 0, 0)$, so

$$\begin{aligned}\kappa &= \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{2(a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4)^{1/2}}{(a^2 + 4b^2t^2 + 9c^2t^4)^{3/2}},\\ \tau &= \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2} = \frac{3abc}{a^2b^2 + 9a^2c^2t^2 + 9b^2c^2t^4}.\end{aligned}$$

Exercise 05.3: Exercise 1, page 18.

Answer Compute $|\alpha''(s)|$.

Exercise 05.4: Exercise 2, page 18.

Solution (b) By direct calculation, we have $\alpha'(t) = (1, \sinh t)$ and $\alpha''(t) = (0, \cosh t)$. The speed is $v = |\alpha'| = \sqrt{1 + \sinh^2 t} = \cosh t$. Up to a sign, the unit normal is obtained by rotating $\mathbf{T}$ a quarter turn: $\mathbf{N} = (-\sinh t, 1)$. Since $\alpha'' = v'\mathbf{T} + \kappa v^2\mathbf{N}$, we have

$$\kappa = \frac{|\alpha'' \cdot \mathbf{N}|}{v^2} = \frac{|(0, \cosh t) \cdot (-\sinh t, 1)|}{\cosh^2 t} = \operatorname{sech} t.$$

(c) The chain rule gives

$$\alpha'(t) = (-3\cos^2 t \sin t, 3\sin^2 t \cos t) = 3\sin t \cos t(-\cos t, \sin t) = 3(\sin^3 t - \sin t, \cos t - \cos^3 t).$$

so $\mathbf{T} = (-\cos t, \sin t)$, $v = 3\sin t\cos t$, and $\mathbf{N} = -(\sin t, \cos t)$. Differentiating again,

$$\alpha''(t) = 3(3\sin^2 t\cos t - \cos t, -\sin t + 3\cos^2 t \sin t) = 3((3\sin^2 t - 1)\cos t, (3\cos^2 t - 1)\sin t).$$

As in part (b),

$$\kappa = \frac{|\alpha'' \cdot \mathbf{N}|}{v^2} = \frac{3\cos t \sin t((3\sin^2 t - 1) + (3\cos^2 t - 1))}{(3\sin t\cos t)^2} = \frac{1}{3\sin t \cos t}.$$

Exercise 05.5: Exercise 3, page 18.

Solution (e) We have $\alpha'(t) = (\sinh t, \cosh t, 1)$, so the speed is

$$|\alpha'(t)| = \sqrt{\cosh^2 t + \sinh^2 t + 1} = \sqrt{2} \cosh t,$$

and therefore $\mathbf{T}(t) = \frac{1}{\sqrt{2}}(\tanh t, 1, \operatorname{sech} t)$.

Next, $\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(\operatorname{sech}^2 t, 0, -\operatorname{sech} t \tanh t) = \frac{1}{\sqrt{2}} \operatorname{sech} t(\operatorname{sech} t, 0, -\tanh t)$. Since the vector part has squared magnitude $\operatorname{sech}^2 t + \tanh^2 t = 1$, we have $\mathbf{N} = (\operatorname{sech} t, 0, -\tanh t)$ and thus $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{2}}(-\tanh t, 1, -\operatorname{sech} t)$.

The curvature and torsion may be calculated using the formulas from Exercise 1 above:

$$\kappa = \frac{\sqrt{2} \cosh t}{(\sqrt{2} \cosh t)^3} = \tfrac{1}{2} \operatorname{sech}^2 t, \qquad \tau = \frac{1}{(\sqrt{2} \cosh t)^2} = \tfrac{1}{2} \operatorname{sech}^2 t.$$

Exercise 05.6: Exercise 4, page 18.

Solution The graph may be parametrized by $\alpha(t) = (t, f(t))$, for which $\alpha'(t) = (1, f'(t))$ and $\alpha''(t) = (0, f''(t))$, and therefore $v = \sqrt{1 + f'(t)^2}$. The scalar cross product $\alpha' \times \alpha''$ is $f''(t)$, so the curvature is

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}.$$