College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 02
Space Paths, Arc Length

The definitions of velocity, speed, acceleration, and regularity make sense for paths in $\mathbf{R}^n$. We will mostly consider paths in space ($\mathbf{R}^3$).

If our path α is regular on some interval containing t_0, we define its *arc length function* to be the integral of the speed,

$$s(t) = \int_{t_0}^{t} |\alpha'(u)|\, du.$$

(If t_0 is unspecified, assume it is 0.) The *arc length* over $[a, b]$ is $s(b) - s(a)$, the definite integral of the speed.

Exercise 02.1: Set up (but *do not attempt to evaluate* the arc length functions for the indicated paths.

(a) $\alpha(t) = (at, bt^2)$; (b) $\alpha(t) = (a\cos t, b\sin t)$; (c) $\alpha(t) = (a\cosh t, b\sinh t)$.

Answer (a) $\displaystyle\int_0^t \sqrt{a^2 + 4b^2u^2}\, du$; (b) $\displaystyle\int_0^t \sqrt{a^2\sin^2 u + b^2\cos^2 u}\, du$; (c) Same as (b) with hyperbolic functions.

Exercise 02.2: Let $r > 0$ and k be real. Calculate the arc length function of the *helix*

$$\alpha(t) = (r\cos t, r\sin t, kt),$$

and give an arc length parametrization.

Answer The speed is constant, $\sqrt{r^2 + k^2}$, so the arc length function is $\ell(t) = t\sqrt{r^2 + k^2}$, and an arc length parametrization is obtained by dividing t by the speed.

Exercise 02.3: Let $r_i > 0$ and k_i be real. Calculate the arc length function of the path

$$\alpha(t) = (r_1\cos(k_1t), r_1\sin(k_1t), r_2\cos(k_2t), r_2\sin(k_2t)).$$

Answer The speed is $\sqrt{r_1^2k_1^2 + r_2^2k_2^2}$, so $\ell(t) = t\sqrt{r_1^2k_1^2 + r_2^2k_2^2}$.

Exercise 02.4: Calculate the arc length function of the parabola $\alpha(t) = (t, \frac{1}{2}t^2)$. Hint: $\int \sqrt{1 + t^2}\, dt = \frac{1}{2}\big[t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2})\big]$.

Answer The speed is $v(t) = \sqrt{1 + t^2}$, so the right-hand side of the integral formula is the arc length function.

Exercise 02.6: Suppose I is an interval of real numbers, and that f is a differentiable, real-valued function on I. The *polar graph* $r = f(\theta)$ may be parametrized by setting $r = f(t)$ and $\theta = t$ in polar coordinates:

$$\alpha(t) = (r\cos\theta, r\sin\theta) = f(t)(\cos t, \sin t).$$

Calculate the velocity and speed of γ. Under what conditions on f is the path γ regular? Hint: You can crunch out the speed by brute force, but if you use vectors and dot products you may save yourself some work!

Solution Using the product rule on the factored form,

$$\alpha'(t) = f'(t)(\cos t, \sin t) + f(t)(-\sin t, \cos t).$$

The two vectors in this derivative expression are an orthonormal pair (!), so the speed is $\|\alpha'(t)\| = \sqrt{f(t)^2 + f'(t)^2}$. The path is regular unless $f(t) = f'(t) = 0$ for some t.

Exercise 02.7: Let R be a positive real number, and let α be the parametrization of the polar graph $r = 2R\cos\theta$.

(a) Find a Cartesian equation for the image of γ. Describe the polar graph geometrically. Suggestion: Multiply both sides by r, then convert to Cartesian.

(b) Use your formulas from the preceding question to calculate the velocity and speed of γ. Is anything noteworthy?

Answer (a) $(x-R)^2+y^2 = R^2$, the circle with center $(R, 0)$ that passes through the origin.

(b) The speed is R, which is both constant, and the same as the speed of the circle $(R\cos t, R\sin t)$ centered at the origin.

Exercise 02.8: Calculate the arc length of the polar graph $r = 1 + \cos\theta$.
Hint: $\sqrt{2 + 2\cos\theta} = 2\cos\frac{\theta}{2}$ by the double-angle formula for cosine.

Answer Since $f(\theta) = 1 + \cos\theta$, the speed of this path is $\sqrt{2 + 2\cos\theta}$. The arc length over $0 \leq \theta \leq 2\pi$ is is

$$\int_0^{2\pi} 2\cos\tfrac{\theta}{2}\,d\theta = 4\sin\tfrac{\theta}{2}\Big|_0^{2\pi} = 8.$$