College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 01
Plane Paths, Velocity and Acceleration

If I is an interval of real numbers and $\alpha : I \to \mathbf{R}^2$ is a plane path, the *velocity* of α is the derivative α', and the *speed* of α is the magnitude of the velocity, $v = |\alpha'|$. Assuming α is *regular* (has non-vanishing velocity), the *unit tangent field* is the normalized velocity,

$$\mathbf{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \qquad \alpha'(t) = v(t)\mathbf{T}(t).$$

The *acceleration* of α is the second derivative α''.

Exercise 01.1: Let $\alpha(t) = (t, t^2)$, which parametrizes the parabola $y = x^2$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Answer (a) Differentiating and normalizing give

$$\alpha'(t) = (1, 2t), \quad \|\alpha'(t)\| = \sqrt{1+4t^2}, \quad \mathbf{T}(t) = \frac{(1, 2t)}{\sqrt{1+4t^2}}.$$

(b) Differentiating a second time, $\alpha''(t) = (0, 2)$. The tangential component of the acceleration is

$$\big(\alpha''(t)\cdot\mathbf{T}(t)\big)\mathbf{T}(t) = \frac{\big(\alpha''(t)\cdot\alpha'(t)\big)\alpha'(t)}{\|\alpha'(t)\|^2} = \frac{4t(1, 2t)}{1+4t^2}.$$

The normal component is therefore the difference,

$$\alpha''(t) - \big(\alpha''(t)\cdot\mathbf{T}(t)\big)\mathbf{T}(t) = \frac{(-4t, 2)}{1+4t^2}.$$

Exercise 01.2: Suppose a and b are positive real numbers, and let $\alpha(t) = (a\cos t, b\sin t)$, which parametrizes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Answer Proceeding along the same lines as the preceding question,

$$\alpha'(t) = (-a\sin t, b\cos t), \quad \|\alpha'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}, \quad \mathbf{T}(t) = \frac{(-a\sin t, b\cos t)}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}.$$

The acceleration is $\alpha''(t) = (-a\cos t, -b\sin t)$; its tangential and normal components may be found as before. The tangential component is

$$\big(\alpha''(t) \cdot \mathbf{T}(t)\big)\mathbf{T}(t) = \frac{(a^2 - b^2)\cos t\sin t(-a\sin t, b\cos t)}{a^2\sin^2 t + b^2\cos^2 t},$$

and the normal component is the difference, which simplifies to

$$\alpha''(t) - \big(\alpha''(t) \cdot \mathbf{T}(t)\big)\mathbf{T}(t) = \frac{-ab(b\cos t, a\sin t)}{a^2\sin^2 t + b^2\cos^2 t}.$$

Exercise 01.3: Suppose a and b are positive real numbers, and let $\alpha(t) = (a\cosh t, b\sinh t)$, which parametrizes one branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Answer Following the preceding question, $\alpha'(t) = (a\sinh t, b\cosh t)$,

$$\|\alpha'(t)\| = \sqrt{a^2\sinh^2 t + b^2\cosh^2 t}, \quad \mathbf{T}(t) = \frac{(a\sinh t, b\cosh t)}{\sqrt{a^2\sinh^2 t + b^2\cosh^2 t}}.$$

The acceleration is $\alpha''(t) = (a\cosh t, b\sinh t)$; its tangential and normal components are

$$\big(\alpha''(t) \cdot \mathbf{T}(t)\big)\mathbf{T}(t) = \frac{(a^2 + b^2)\cosh t\sinh t\,(a\sinh t, b\cosh t)}{a^2\sinh^2 t + b^2\cosh^2 t},$$

and

$$\alpha''(t) - \big(\alpha''(t) \cdot \mathbf{T}(t)\big)\mathbf{T}(t) = \frac{ab(b\cosh t, a\sinh t)}{a^2\sinh^2 t + b^2\cosh^2 t}.$$

Exercise 01.5: Assume α is a regular path on some interval I, and write $\alpha'(t) = v(t)\mathbf{T}(t)$.

(a) Calculate the acceleration α'' using the product rule. Using Exercise 01.4, show one summand is proportional to $\mathbf{T}$ and the other orthogonal to $\mathbf{T}$. Show $\alpha''(t) \cdot \mathbf{T}(t) = v'(t)$.

The components $v'(t)\mathbf{T}(t)$ and $\alpha''(t) - v'(t)\mathbf{T}(t)$ are called the *tangential component* and the *normal component* of the acceleration. You may want to verify these are what you found in the (b) parts of earlier exercises.

Solution By the product rule,

$$\alpha''(t) = \frac{d}{dt}\big[v(t)\mathbf{T}(t)\big] = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t).$$

The first summand is obviously a scalar multiple of $\mathbf{T}(t)$, and the second is perpendicular to $\mathbf{T}(t)$ by the fundamental idiom, which guarantees $\mathbf{T} \cdot \mathbf{T}' = 0$. Direct calculation gives $\alpha''(t) \cdot \mathbf{T}(t) = \mathbf{v}'(t)\big(\mathbf{T}(t) \cdot \mathbf{T}(t)\big)$, which is $v'(t)$ since $\mathbf{T}(t)$ is a unit vector.

(b) **Erratum**: The definition of curvature here is only correct when α is parametrized by arc length. As shown on page 13 of the book, we have $\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$ for a general regular path, i.e., not necessarily parametrized by arc length.

If the normal component of acceleration is non-zero, its magnitude $\kappa(t)$ is called the *unsigned curvature* of the path, and the corresponding unit vector $\mathbf{N}(t)$ is called the *principal normal*. Show that

$$\alpha''(t) = v'(t)\mathbf{T}(t) + \kappa(t)\mathbf{N}(t).$$

(Geometrically, the tangential component measures how the speed of α is changing and the normal component measures how the direction of α is changing.)

Solution By definition, we have $\kappa(t) = v(t)\|\mathbf{T}'(t)\|$ and $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$; substituting, the normal component of the acceleration is

$$v(t)\mathbf{T}'(t) = v(t)\|\mathbf{T}'(t)\| \,\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \kappa(t)\mathbf{N}(t).$$

(c) Assume $R > 0$, $v \neq 0$, and x_0, y_0 are real numbers. Calculate the curvature and principal normal for the circle $\alpha(t) = (x_0 + R\cos(vt/R), y_0 + R\sin(vt/R))$ of radius R and speed $|v|$ centered at (x_0, y_0).

Answer The acceleration $\alpha''(t) = -\dfrac{v^2}{R}\big(\cos(vt/R), \sin(vt/R)\big)$ is normal to the velocity because the speed is constant. To find the curvature, we assume our circle is parametrized by arc length, i.e., $v = 1$, for which $\kappa = 1/R$.

(d) Calculate the curvature of the parabola in the first exercise.

Answer By Exercise 01.1, the normal component of the acceleration of the parabola is

$$\frac{2(-2t, 1)}{1 + 4t^2} = \frac{2}{\sqrt{1 + 4t^2}}\,\frac{(-2t, 1)}{\sqrt{1 + 4t^2}};$$

the scalar is $v(t)\kappa(t)$ since the vector is unit. The parabola therefore has curvature

$$\kappa(t) = \frac{2}{1 + 4t^2}.$$