

**College of the Holy Cross, Fall Semester, 2021**  
**Math 302** (Professor Hwang), Meeting 05  
**The Frenet Equations**

**Exercise 05.1:** (Linear algebra warm-up) Suppose  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$  is an ordered orthonormal triple in  $\mathbf{R}^3$ .

- (a) Prove the set  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  is linearly independent. That is, if  $a\mathbf{T} + b\mathbf{N} + c\mathbf{B} = \mathbf{0}$  for some real  $a, b, c$ , then  $a = b = c = 0$ .
- (b) Prove the set  $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$  spans  $\mathbf{R}^3$ . That is, every vector  $v$  can be written as a linear combination. Hint: If  $v = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}$ , dot each side with the orthonormal elements in turn to express  $a, b$ , and  $c$  in terms of  $v$ . Then show that the resulting components do in fact express  $v$  as a linear combination.
- (c) Let  $I$  be an open interval of real numbers, and let  $f : I \rightarrow \mathbf{R}^3$  be a smooth mapping. Prove that the image of  $f$  lies on a sphere centered at the origin if and only if  $f \cdot f' = 0$ .

**Exercise 05.2:** Suppose  $\alpha$  is a regular path in  $\mathbf{R}^3$ .

- (a) Prove that the curvature  $\kappa$  is given by  $\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$ . Hints: If  $v$  denotes the speed, then  $\alpha' = v\mathbf{T}$  and  $\alpha'' = v'\mathbf{T} + \kappa v^2\mathbf{N}$ .
- (b) Prove that the torsion  $\tau$  is given by  $\tau = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2}$ .
- (c) Calculate the curvature and torsion of the twisted cubic  $\alpha(t) = (at, bt^2, ct^3)$ .

**Exercise 05.3:** Exercise 1, page 18.

**Exercise 05.4:** Exercise 2, page 18.

**Exercise 05.5:** Exercise 3, page 18.

**Exercise 05.6:** Exercise 4, page 18.