College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 04
The Frenet-Serret Frame

Let I be an interval of real numbers. Recall that a path $\alpha : I \to \mathbf{R}^3$ is *smooth* if all its derivatives exist, is *regular* if in addition the velocity is never zero, and is *parametrized by arc length* if $|\alpha'(s)| = 1$ for all s, so that the distance along the path is precisely the change in s.

We say α is a *plane path* if there exists a plane in $\mathbf{R}^3$ containing the image of α. In practice this often means the third component of α is identically 0, in which case we usually write $\alpha(t) = \big(x(t), y(t)\big)$.

Exercise 04.1: Let a and b be positive real numbers. Calculate the unit tangent field $\mathbf{T}$ and the unit normal field of the ellipse $\alpha(t) = (a \cos t, b \sin t)$. Decompose the acceleration α'' into tangential and normal components. Compare your result with the interactive demo.

Exercise 04.2: Let a and b be positive real numbers. Calculate the unit tangent field $\mathbf{T}$ and the unit normal field of the hyperbola $\alpha(t) = (a \cosh t, b \sinh t)$. Decompose the acceleration α'' into tangential and normal components. Compare your result with the interactive demo.

Exercise 04.3: Let $a > 0$ be a positive real number. Calculate the unit tangent field $\mathbf{T}$ and the unit normal field of the astroid $\alpha(t) = (a \cos^3 t, a \sin^3 t)$. Decompose the acceleration α'' into tangential and normal components. Compare your result with the interactive demo.

Exercise 04.4: Let $r > 0$, $v > 0$, and k be real numbers. Calculate the unit tangent field $\mathbf{T}$, the principal normal field $\mathbf{N}$, and the binormal field $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ of the circular helix $\alpha(t) = \big(r \cos(vt), r \sin(vt), kvt\big)$. Compare your result with the interactive demo.

Under what conditions is α parametrized by arc length?

Exercise 04.5: Let a, b, and c be real numbers. Calculate the unit tangent field $\mathbf{T}$, the principal normal field $\mathbf{N}$, and the binormal field of the twisted cubic $\alpha(t) = (at, bt^2, ct^3)$. Compare your result with the interactive demo.