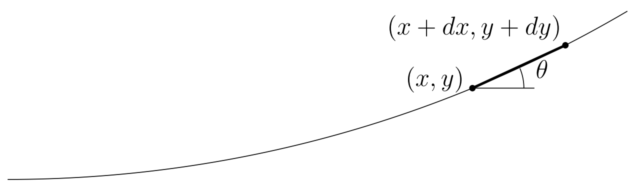
College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 03
Differential Equations and the Shape of Cables

When a suspension bridge is built, flexible cables are hung between two towers. Under their own weight, what shape do the cables assume? As the deck is attached, the primary vertical force on the cables becomes the weight of the deck, which far outweighs the cables. If the deck has constant mass per unit of horizontal distance, what shape do the cables assume under load?

Today we'll use differential equations to answer both questions. In each case, the cables' shapes are important, well-known curves. A cable hanging under its own weight makes a special type of *catenary*, a scaled hyperbolic cosine graph. An evenly-loaded cable, by contrast, describes a parabola.

The model for each case is a graph $y = f(x)$. We'll view the graph as comprising infinitesimal segments between (x, y) and $(x + dx, y + dy)$, and making angle $\theta = \theta(x)$ with the horizontal. There are three forces acting on the segment: A tension of magnitude $T(x)$



pulling to the left, a tension $T(x + dx)$ pulling to the right, and the weight due to gravity pulling vertically down. Because the segment is in equilibrium, these forces add to zero. Resolving the horizontal and vertical components will give our differential equations.

The tension at the left end has components $-T(x)(\cos\theta, \sin\theta)$. At the right end, we have $\theta(x + dx) = \theta + d\theta$; linear approximations from calculus give

$$\begin{aligned}\cos(\theta + d\theta) &= \cos\theta - \sin\theta\, d\theta,\\ \sin(\theta + d\theta) &= \sin\theta + \cos\theta\, d\theta,\end{aligned}$$

so the components of the tension are

$$T(x + dx)\big[(\cos\theta, \sin\theta) + (-\sin\theta, \cos\theta)\, d\theta\big].$$

For the moment, we'll denote the vertical force of weight as $-dw$. The horizontal component of force on the segment is

$$0 = -T(x)\cos\theta + T(x + dx)\cos\theta(x + dx); \quad \text{thus } T(x)\cos\theta = T(x + dx)\cos\theta(x + dx).$$

We conclude that the function $T(x)\cos\theta(x)$ is constant, say T_0. Differentiating gives the relation $T'(x)\cos\theta\, dx - T(x)\sin\theta\, d\theta = 0$, or $T'(x)\cos\theta\, dx = T(x)\sin\theta\, d\theta$.

The vertical component of force on the segment is

$$\begin{aligned}0 &= -T(x)\sin\theta + T(x + dx)\sin\theta(x + dx) - dw\\ &= -T(x)\sin\theta + [T(x) + T'(x)\, dx][\sin\theta + \cos\theta\, d\theta] - dw\\ &= T(x)\cos\theta\, d\theta + T'(x)\sin\theta\, dx - dw \quad \text{discarding the second-order differential.}\end{aligned}$$

To summarize, our graph $y = f(x)$ satisfies $T(x)\cos\theta = T_0$, $T'(x)\cos\theta\,dx = T(x)\sin\theta\,d\theta$. Multiplying the second by $\tan\theta$ and substituting gives

$$\begin{aligned}0 &= T(x)\cos\theta\,d\theta + T'(x)\sin\theta\,dx - dw\\ &= T(x)\left[\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right]d\theta - dw\\ &= T(x)\sec\theta\,d\theta - dw.\end{aligned}$$

Exercise 03.1: When a cable hangs under its own weight, the downward force is proportional to the arc length: $dw = k\,ds$.

(a) Show that $\tan\theta = f'(x) = \frac{dy}{dx}$, and conclude that $ds = \sec\theta\,dx$.

(b) Separate variables and integrate to obtain a non-differential equation for θ and x. Hints: $\int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$, and by choosing coordinates with the origin below the lowest point on the cable we may assume $\theta = 0$ when $x = 0$.

(c) Show that the equation you found in (b) implies $ds = \cosh x\,dx$. Express this as a differential equation in x and y, and integrate to find $y = f(x)$.

Exercise 03.2: When a cable hangs under uniform load much larger than its own weight, the downward force is proportional to the width: $dw = k\,dx$. Solve the resulting differential equation.

Exercise 03.3: Work out Example 2 on page 5 of the book yourself.

Exercise 03.4: Exercise 10 on page 9.