College of the Holy Cross, Fall Semester, 2021
Math 302 (Professor Hwang), Meeting 01
Plane Paths, Velocity and Acceleration

If I is an interval of real numbers and $\alpha : I \to \mathbf{R}^2$ is a plane path, the *velocity* of α is the derivative α', and the *speed* of α is the magnitude of the velocity, $v = |\alpha'|$. Assuming α is *regular* (has non-vanishing velocity), the *unit tangent field* is the normalized velocity,

$$\mathbf{T}(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \qquad \alpha'(t) = v(t)\mathbf{T}(t).$$

The *acceleration* of α is the second derivative α''.

Exercise 01.1: Let $\alpha(t) = (t, t^2)$, which parametrizes the parabola $y = x^2$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Exercise 01.2: Suppose a and b are positive real numbers, and let $\alpha(t) = (a \cos t, b \sin t)$, which parametrizes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Exercise 01.3: (This question refers to the hyperbolic functions, which are defined in Exercise 01.6 below, and obey identities similar to the circular functions but without all the vexing signs.) Suppose a and b are positive real numbers, and let $\alpha(t) = (a \cosh t, b \sinh t)$, which parametrizes one branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(a) Calculate the velocity, the speed, and the unit tangent field $\mathbf{T}$.

(b) Calculate the acceleration, and write the acceleration as the sum of a vector parallel to $\mathbf{T}$ and a vector orthogonal to $\mathbf{T}$.

Exercise 01.4: Assume $\mathbf{F}$ is a differentiable vector-valued function defined on some interval I, and that $|\mathbf{F}(t)|^2 = \mathbf{F}(t) \cdot \mathbf{F}(t)$ is constant. (Geometrically, $\mathbf{F}$ is a path on some sphere centered at the origin.)

Show that $\mathbf{F}(t) \cdot \mathbf{F}'(t) = 0$ for all t, and interpret this geometrically. (We will call this fact the *fundamental idiom of differential geometry*.)

Exercise 01.5: Assume α is a regular path on some interval I, and write $\alpha'(t) = v(t)\mathbf{T}(t)$.

(a) Calculate the acceleration α'' using the product rule. Using Exercise 01.4, show one summand is proportional to $\mathbf{T}$ and the other orthogonal to $\mathbf{T}$. Show $\alpha''(t) \cdot \mathbf{T}(t) = v'(t)$.

The components $v'(t)\mathbf{T}(t)$ and $\alpha''(t) - v'(t)\mathbf{T}(t)$ are called the *tangential component* and the *normal component* of the acceleration. You may want to verify these are what you found in the (b) parts of earlier exercises.

(b) If the normal component of acceleration is non-zero, its magnitude $\kappa(t)$ is called the *unsigned curvature* of the path, and the corresponding unit vector $\mathbf{N}(t)$ is called the *principal normal.* Show that

$$\alpha''(t) = v'(t)\mathbf{T}(t) + \kappa(t)\mathbf{N}(t).$$

(Geometrically, the tangential component measures how the speed of α is changing and the normal component measures how the direction of α is changing.)

(c) Assume $R > 0$, $v \neq 0$, and x_0, y_0 are real numbers. Calculate the curvature and principal normal for the circle $\alpha(t) = (x_0 + R\cos(vt/R), y_0 + R\sin(vt/R))$ of radius R and speed $|v|$ centered at (x_0, y_0).

(d) Calculate the curvature of the parabola in the first exercise.

Exercise 01.6: The functions $\cosh t = \frac{1}{2}(e^t + e^{-t})$ and $\sinh t = \frac{1}{2}(e^t - e^{-t})$ are called the *hyperbolic cosine* and *hyperbolic sine.* (We'll use these throughout the course, so please keep these definitions and the properties below in mind.)

(a) On a single set of axes, carefully sketch the graphs $y = \frac{1}{2}e^t$, $y = \frac{1}{2}e^{-t}$, $y = \cosh t$, and $y = \sinh t$.

Establish the following properties:

(b) $\cosh(-t) = \cosh t$ and $\sinh(-t) = -\sinh t$ for all real t.

(c) $\cosh t + \sinh t = e^t$ and $\cosh t - \sinh t = e^{-t}$ for all real t.

(d) $\cosh^2 t - \sinh^2 t = 1$ for all real t.

(e) $\cosh^2 t + \sinh^2 = \cosh(2t)$ for all real t.

(f) $\cosh' = \sinh$, $\sinh' = \cosh$.

Exercise 01.7: The functions

$$\tanh t = \frac{\cosh t}{\sinh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad \operatorname{sech} t = \frac{1}{\cosh t},$$

are called the *hyperbolic tangent* and *hyperbolic secant.*

(a) Show that $\tanh' = \operatorname{sech}^2$ and $\operatorname{sech}' = \operatorname{sech} \cdot \tanh$.

(b) Show that tanh is a bijection from $\mathbf{R}$ to the open interval $(-1, 1)$.