College of the Holy Cross, Spring 2016
Math 244 Review Sheet for Midterm 3

The final will be held in the usual classroom on Thursday, May 12, 3:00–5:30 PM. The exam covers all the material in the course. The questions below range from routine practice to challenging. Do not confine your studying only to this sheet; review material from earlier in the course as needed.

Review Questions

Throughout, the standard basis of $\mathbf{R}^n$ is $(\boldsymbol{e}_j)_{j=1}^n$, and the standard basis of P_n is $(t^k)_{k=0}^n$.

1. Write down a 3×4 matrix A' in reduced row-echelon form, and perform five or six elementary row operations to get a "scrambled" matrix A. Write down a particular 4×1 column $\boldsymbol{x}$, and calculate $\boldsymbol{b} = A\boldsymbol{x}$.

   Exchange "A" and "$\boldsymbol{b}$" matrices with a classmate. (Keep copies of A' and $\boldsymbol{x}$; these are your friend's "answer key".)

   (a) Calculate the rank and nullity of A by putting A in reduced row-echelon form.

   (b) Find bases for the null space of A and the column space of A.

   (c) Find all solutions of the system $A\boldsymbol{x} = \boldsymbol{b}$ for the A and $\boldsymbol{b}$ you were given.

   (d) Swap answers with your friend; resolve any discrepencies.

   (e) Repeat as many times as desired.

2. Find an ordered basis $(\boldsymbol{v}_j)_{j=1}^3$ of $\mathbf{R}^3$. (For example, perform elementary row operations on the identity matrix until you have a reasonably scrambled matrix, P, and let $\boldsymbol{v}_j$ be the jth column of P.

   Use the row-reduction algorithm from Chapter 1 to calculate the inverse matrix P^{-1}.

   Pick three integers $(\lambda^j)_{j=1}^3$ in non-increaing order, not necessarily distinct but not all equal. (For example, 2, 2, -1 is all right, but 2, 1, 2 isn't non-decreasing, and -1, -1, -1 are all equal.) Let $D = \operatorname{diag}[\lambda^1, \lambda^2, \lambda^3]$, and calculate the product $A = PDP^{-1}$.

   Exchange "A" matrices with a classmate. (Keep copies of D, P, and P^{-1}; these are your friend's "answer key".)

   (a) Calculate the characteristic polynomial and eigenvalues of A.

   (b) For each eigenvalue of A, find a basis of the corresponding eigenspace.

   (c) Find the matrix P that diagonalizes A so that the eigenvalues are in non-decreasing order.

   (d) Swap answers with your friend; resolve any discrepencies.

   (e) Repeat as many times as desired.

3. For each matrix, either diagonalize or prove the matrix is not diagonalizable

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4. Find *orthogonal* matrices that diagonalize each of the following:

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 2 & 2 \\ 2 & 6 & 4 \\ 2 & 4 & 6 \end{bmatrix}.$$

5. Let $\boldsymbol{x} \in \mathbf{R}^n$ be non-zero, and let A be the outer product $\boldsymbol{x}\boldsymbol{x}^\mathsf{T}$.

   (a) Find the eigenvalues and eigenspaces of A, and show A is diagonalizable.

   (b) If c is a real scalar, find the eigenvalues and corresponding eigenspaces of $cI_n + \boldsymbol{x}\boldsymbol{x}^\mathsf{T}$.

6. An $n \times n$ real matrix A is *nilpotent* if there exists a positive integer k such that $A^k = \mathbf{0}^{n\times n}$. Prove that if A is nilpotent, then

   (a) 0 is the only eigenvalue of A. (b) A is diagonalizable if and only if $A = \mathbf{0}^{n\times n}$.

7. Let $L : \mathbf{R}^n \to \mathbf{R}^n$ be the "left-shift" operator

$$L(x^1, x^2, x^3, \dots, x^n) = (x^2, x^3, \dots, x^n, 0).$$

   (a) Describe the kernel and image of L, and give bases for each.

   (b) Find the standard matrix A of L, and describe the null space and column space of A. Be sure your answer is consistent with (a).

   (c) Show that $L^{n-1} \neq 0$ (the $(n-1)$-fold composition), but $L^n = 0$. (Consequently, A is nilpotent, see preceding question.) Describe the matrices A^k for $1 < k < n$.

8. Let A be an $n \times n$ real skew-symmetric matrix. Prove 0 is the only real eigenvalue of A.

9. Let $(V, +, \cdot)$ be a finite-dimensional vector space, $T : V \to V$ a diagonalizable linear operator. Prove that if $W \subseteq V$ is T-invariant, i.e., if $T(W) \subseteq W$, and if $T_W : W \to W$ is the induced operator, then λ is an eigenvalue of T_W if and only if $W \cap E_\lambda \neq \{0\}$, and the λ-eigenspace of T_W is $W \cap E_\lambda$.

10. If A is an $n \times n$ real matrix and t is real, we define

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I_n + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \cdots.$$

   If a, b, and c denote real numbers, calculate $\exp(tA)$ for the following A:

$$\operatorname{diag}[a, b, c], \qquad \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \qquad \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}, \qquad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \qquad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

   (Hint: If $A = PDP^{-1}$, then $\exp(tA) = P\exp(tD)P^{-1}$, and $\exp(tD)$ is found easily. It may help to use the result of the next question.)

11. Let A and B be *commuting* real $n \times n$ matrices. Prove:

   (a) $(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k$.

   (b) $\exp[t(A + B)] = \exp(tA)\exp(tB)$. Hint: Multiply the power series on the right, gather terms of the same degree in t, and use the binomial theorem of part (a) to simplify the sum of these terms.