College of the Holy Cross, Spring 2016
Math 244 Review Sheet for Midterm 3

The third midterm will be held in class on Friday, April 22. The test covers the material up through the end of Chapter 4, including Gram-Schmidt, orthogonal projection, determinants, the kernel and image of a linear transformation, the space of linear transformations, the matrix of a linear transformation, change of basis, the null space and column space of a matrix, and the Rank-Nullity Theorem in various forms.

Review Questions

Throughout, the standard basis of $\mathbf{R}^n$ is $(\boldsymbol{e}_j)_{j=1}^n$, and the standard basis of P_n is $(t^k)_{k=0}^n$. Any unassigned exercises from Chapters 3 and 4 are good practice.

1. Consider the real matrices $A = \begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

   (a) Find all real λ for which $A - \lambda I_3$ is *not* invertible. (These numbers are the so-called "eigenvalues" of A.)

   (b) For each λ in part (a), find a basis of the null space of $A - \lambda I_3$.

   (c) Repeat parts (a) and (b) for B.

2. Let $J = \mathrm{Rot}_{\pi/2} : \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by a quarter turn.

   (a) Find the standard matrix A of J, and verify that $A^2 = -I_2$ (by multiplying matrices). Interpret this result geometrically.

   (b) If $\boldsymbol{v}_1 = (1, 1)$ and $\boldsymbol{v}_2 = (1, 2)$, and if $S' = (\boldsymbol{v}_1, \boldsymbol{v}_2)$, find the matrix B of J with respect to S'. Is it true that $B^2 = -I_2$? Explain. (Suggestion: To find $[J]_{S'}^{S'}$, find the change of basis matrix $[I]_{S'}^{S}$ and its inverse, and conjugate the standard matrix of J.)

3. Let A be a real $m \times n$ matrix, let $T = \mu_A : \mathbf{R}^n \to \mathbf{R}^m$ be the associated linear transformation, and consider the function $B : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by

   $$B(\boldsymbol{x}, \boldsymbol{y}) = \langle T(\boldsymbol{x}), T(\boldsymbol{y}) \rangle.$$

   (a) Show that B is symmetric and bilinear, and that $B(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$.

   (b) Show B positive-definite is and only if A is invertible, if and only if T is injective.

4. Let A and B be real $n \times n$ matrices, and assume A and B are similar, i.e., there exists an invertible matrix P such that $B = P^{-1}AP$.

   (a) Show that $\det B = \det A$.

   (b) Suppose $(V, +, \cdot)$ is finite-dimensional $T : V \to V$ is a linear transformation. If S and S' are bases of V, show that $A = [T]_S^S$ and $B = [T]_{S'}^{S'}$ have the same determinant.

   (Consequently, we may define $\det T = \det[T]_S^S$ for any convenient basis S of V.)

5. Show that the function $\langle A, B\rangle = \mathrm{tr}(A^{\mathsf{T}}B)$ is an inner product on the space $\mathbf{R}^{m\times n}$ of real $m \times n$ matrices. (Hint: The direct approach works if all else fails. If $A = [A^i_j]$ and $B = [B^k_\ell]$, calculate $\langle A, B\rangle$ in terms of the entries. Looking at 2×3 matrices may be helpful to start.)

6. Let $\boldsymbol{v} \in \mathbf{R}^n$ and $\boldsymbol{w} \in \mathbf{R}^m$ be non-zero vectors, and define $T : \mathbf{R}^n \to \mathbf{R}^m$ by

$$T(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{v}\rangle \boldsymbol{w}.$$

   (a) Show T is linear.

   (b) Show that $\ker T$ is the subspace orthogonal to $\boldsymbol{v}$.

   (c) Show that $(\boldsymbol{w})$ is a basis for $\mathrm{im}\, T$.

   (d) If $\boldsymbol{v} = (1, -1, 2, 4)$ and $\boldsymbol{w} = (3, -1, 5)$, calculate the standard matrix of T, and use row operations to find bases of $\ker T$ and $\mathrm{im}\, T$. Be sure your results are consistent with parts (b) and (c).

7. Define the mapping $T : P_2 \to \mathbf{R}^2$ by $T(p) = \begin{bmatrix} p(-1) \\ p(2) \end{bmatrix}$.

   (a) Show T is linear.

   (b) Find the matrix of T with respect to the standard bases.

   (c) Find bases for $\ker T$ and $\mathrm{im}\, T$, and verify the Rank-Nullity Theorem.

8. Consider the linear mappings Sym and Skew : $\mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ defined by

$$\mathrm{Sym}(A) = \tfrac{1}{2}(A + A^{\mathsf{T}}), \qquad \mathrm{Skew}(A) = \tfrac{1}{2}(A - A^{\mathsf{T}}).$$

   Find the matrices of these transformations with respect to:

   (a) The basis $(\boldsymbol{e}^1_1, \boldsymbol{e}^2_2, \boldsymbol{e}^2_1, \boldsymbol{e}^1_2)$.

   (b) The basis $(\boldsymbol{e}^1_1, \boldsymbol{e}^2_2, \boldsymbol{e}^2_1 + \boldsymbol{e}^1_2, \boldsymbol{e}^2_1 - \boldsymbol{e}^1_2)$.

   (c) Verify the change of basis formula in Corollary 4.37 (ii) (p. 83).

9. Let $V \subseteq \mathcal{C}^\infty$ be the plane spanned by the functions $e_1(x) = e^x$ and $e_{-1}(x) = e^{-x}$, and let D denote differentiation.

   (a) Show that D maps V to V, i.e., that $D(V) \subseteq V$.

   (b) Find the matrix of D with respect to the basis $S = (e_1, e_2)$.

   (c) Let $S' = (\cosh, \sinh)$, with $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ the *hyperbolic cosine* and *hyperbolic sine*. Show that S' is a basis of V, and calculate the change of basis matrices $[I_V]^{S'}_S$ and $[I_V]^S_{S'}$.

   (d) Calculate $[D]^{S'}_{S'}$ in two ways: Directly (using the definition), and using Corollary 4.37 (page 83).

10. Let $(V, +, \cdot)$ denote the space of linear, real-valued functions on $\mathbf{R}^n$. That is, $\lambda \in V$ if and only if $\lambda : \mathbf{R}^n \to \mathbf{R}$ is linear.

   (a) Show that if $\lambda \in V$, then λ is uniquely determined by the n real numbers $\lambda_i = \lambda(\boldsymbol{e}_i)$.

   (b) Show that V is isomorphic as a vector space to $(\mathbf{R}^n)^*$, the space of row matrices.