College of the Holy Cross, Spring 2016
Math 244 Review Sheet for Midterm 2

The second midterm will be held in class on Wednesday, March 23. The test covers the material up through orthonormal sets in Section 3.2.

Review Questions

1. Find a basis for the space of 2×2 real matrices that commute with $\boldsymbol{e}_1^2$.

2. Let A be an $n \times n$ real matrix ($n \geq 2$), $W = \{B \text{ in } \mathbf{R}^{n\times n} : AB = BA\}$. Prove W is a subspace of $(\mathbf{R}^{n\times n}, +, \cdot)$ of dimension at least 2.

3. Let A be an $n \times n$ real matrix. Prove the columns of A form a linearly independent set in $\mathbf{R}^n$ if and only if A is invertible.

4. Let A be an $m \times n$ real matrix with $m < n$.

   (a) Show that the solution space of $A\boldsymbol{x} = \mathbf{0}$ has dimension $\geq n - m$. Hint: How many leading 1's can the reduced row-echelon form of A have?

   (b) If the rows of A are a linearly independent set in $(\mathbf{R}^n)^*$, what can you say about the reduced row-echelon form of A? Is the converse true?

   (c) Under what conditions do the columns of A form a linearly independent subset of $\mathbf{R}^m$? A spanning set?

5. Let $W = \{A \text{ in } \mathbf{R}^{2\times 2} : \det A = 0\}$.

   (a) Is W closed under addition?

   (b) Is W closed under scalar multiplication?

6. (In this question, superscripts denote exponentiation.) Suppose $a < b < c$ are real numbers.

   (a) Prove that the matrix $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is invertible.

   (b) Prove that the set $(1, t, t^2)$ is linearly independent.

7. A square matrix $A = [A_j^i]$ is *upper triangular* if $A_j^i = 0$ for $i > j$.

   (a) Show that the set U_n of $n \times n$ real upper-triangular matrices is a subspace of $(\mathbf{R}^{n\times n}, +, \cdot)$. Find a basis for U_n; what is $\dim U_n$?

   (b) Analogously, define the concept of a *lower triangular* matrix. (See Definition 2.82, p. 47, for answer.) Show L_n is a subspace of $(\mathbf{R}^{n\times n}, +, \cdot)$, and $\dim L_n = \dim U_n$.

   (c) Verify the dimension theorem for U_n and L_n. That is, calculate the dimensions of $U_n \cap L_n$ and $U_n + L_n$, and show $\dim(U_n + L_n) = \dim U_n + \dim L_n - \dim(U_n \cap L_n)$. (Suggestion: What kind of matrices are in $U_n \cap L_n$?)

8. In the vector space $(\mathbf{R}^{3\times 3}, +, \cdot)$ of real 3×3 matrices, consider the subspaces $X = \mathrm{Sym}^3$ of symmetric matrices and $Y = \{B : \operatorname{tr} B = 0\}$ of trace-free matrices. Verify the dimension theorem. (That is, calculate the dimensions of $X \cap Y$, X, Y, and $X + Y$, and show $\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y)$.)

9. Generalize the preceding question to square matrices of arbitrary size.

10. Suppose $S = (\boldsymbol{v}_j)_{j=1}^n$ and $S' = (\boldsymbol{v}'_i)_{i=1}^n$ are bases of some vector space $(V, +, \cdot)$.

    (a) Show there exist scalars A^i_j, for $i, j = 1, \dots, n$, such that

    $$\boldsymbol{v}_j = \sum_{i=1}^{n} A^i_j \boldsymbol{v}'_i \quad \text{for all } j = 1, \dots, n.$$

    (b) Let $A = [A^i_j]$ be the $n \times n$ matrix whose entries are the scalars in part (a). Prove that if $\boldsymbol{x} \in \mathbf{R}^n$ is arbitrary, then $A[\boldsymbol{x}]^S = [\boldsymbol{x}]^{S'}$. That is, multiplication by A converts the coordinate vector of $\boldsymbol{x}$ with respect to S into the coordinate vector of $\boldsymbol{x}$ with respect to S'. (This is mostly an exercise in unpacking definitions; see Definition 2.60, p. 42.)

11. Let $\boldsymbol{v} = (1, -2, 2)$ and $\boldsymbol{a} = (1, 1, 1)$.

    (a) Write $\boldsymbol{v}$ as the sum of a vector $\boldsymbol{v}_1$ parallel to $\boldsymbol{a}$ and a vector $\boldsymbol{v}_2$ orthogonal to $\boldsymbol{a}$.

    (b) Show by direct calculation that $\|\boldsymbol{v}\|^2 = \|\boldsymbol{v}_1\|^2 + \|\boldsymbol{v}_2\|^2$.

12. Show that the set of vectors

    $$\gamma(t) = (\cos t + \sin t, \cos t - \sin t, \cos t + \sin t, \cos t - \sin t), \qquad t \text{ real},$$

    describes a circle, i.e., a set of points lying in some plane and at a fixed distance from some point. Suggestion: Decompose $\gamma(t)$ into a "trigonometric linear combination" and show the resulting "basis vectors" are orthogonal and have the same length. What are the circle's center and radius?

13. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be arbitrary vectors in $\mathbf{R}^n$. Prove that

    $$\boldsymbol{x} \cdot \boldsymbol{y} = \tfrac{1}{4}\left(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2\right).$$

    Hint: Expand the right-hand side using properties of the dot product.

14. Each part refers to the vector space of continuous, real-valued functions on $[-\pi, \pi]$, equipped with the inner product

    $$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

    Let k and ℓ denote positive integers, and define $c_k(x) = \cos(kx)$ and $s_k = \sin(kx)$.

    (a) Show that the following pairs of functions are orthogonal:
    c_k and s_ℓ; c_k and c_ℓ if $k \neq \ell$; s_k and s_ℓ if $k \neq \ell$.
    Suggestion: The addition formulas for cos and sin,

    $$\left.\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b, \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b, \end{aligned}\right\} \text{imply} \left\{\begin{aligned} \cos a \cos b &= \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right], \\ \sin a \sin b &= \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right], \\ \sin a \cos b &= \tfrac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]. \end{aligned}\right.$$

    (b) Calculate $\|c_k\|^2 = \langle c_k, c_k \rangle$ and $\|s_k\|^2 = \langle s_k, s_k \rangle$.

    (c) Suppose $f = \sum_{k=0}^{n} (a_k c_k + b_k s_k)$ (compare Example 2.14). Calculate $\langle f, f \rangle$. Suggestion: Work out the cases $n = 1$ or 2 if you don't immediately see a pattern.