College of the Holy Cross, Spring 2016
Math 244 Review Sheet for Midterm 1

The first midterm will be held in class on Friday, February 19. The test covers the material up through Section 2.1.

Review Questions

1. Use the row reduction web program to generate a system of equations. Solve by hand on paper, then use the program to solve; be sure you get the same reduced row-echelon matrix each way. Repeat as necessary.

2. Repeat the preceding exercise using the matrix inversion we program.

3. Suppose

$$A = \begin{bmatrix} -1 & 0 & 7 \\ 3 & 2 & -4 \\ -5 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix}.$$

Calculate the indicated matrix if possible, or explain why the expression is undefined:

$$AB, \quad BA, \quad CB + D, \quad A + BC, \quad CD, \quad DC.$$

4. Write out the 3×3 matrices given by the indicated formulas:
(a) $A^i_j = i + j$; (b) $A^i_j = i - j$; (c) $A^i_j = (-1)^{i+j}$; (d) $A^i_j = 2^{i-j}$; (e) $A^i_j = 2^i\,\delta^i_j$.

5. If $A = [A^i_j]$ is an $n \times n$ matrix, the *trace* of A is the number

$$\operatorname{tr} A = A^1_1 + A^2_2 + \cdots + A^n_n = \sum_{k=1}^{n} A^k_k.$$

(a) Calculate the trace of each matrix in the preceding question, and the trace of I_n, the $n \times n$ identity matrix.

(b) Show that if A is $m \times n$ and B is $n \times m$, then $\operatorname{tr}(BA) = \operatorname{tr}(AB)$. (Note that the respective products do not have the same size.)

(c) Show that if A and B are $n \times n$ matrices, then $[A, B] \neq I_n$. Suggestion: Use parts (a) and (b).

(d) Show that if P is invertible $n \times n$ and A is an arbitrary $n \times n$ matrix, then $\operatorname{tr}(P^{-1}AP) = \operatorname{tr} A$. Suggestion: Use part (b).

(e) True or false: $\operatorname{tr}(AB) = (\operatorname{tr} A)(\operatorname{tr} B)$. $\quad \operatorname{tr}(cA) = c \operatorname{tr} A$. $\quad \operatorname{tr} A^{\mathsf{T}} = \operatorname{tr} A$.

6. Let $(\boldsymbol{e}_j)_{j=1}^3$ denote the standard basis of $\mathbf{R}^3$. Find the unique 3×3 matrix A satisfying the given conditions:

(a) $A\boldsymbol{e}_1 = \boldsymbol{e}_2$, $A\boldsymbol{e}_2 = \boldsymbol{e}_3$, and $A\boldsymbol{e}_3 = \boldsymbol{e}_1$. (b) $A\boldsymbol{e}_1 = \boldsymbol{e}_1$, $A\boldsymbol{e}_2 = \boldsymbol{e}_3$, and $A\boldsymbol{e}_3 = \boldsymbol{e}_2$.

(c) $A\boldsymbol{e}_1 = \boldsymbol{e}_1$, $A\boldsymbol{e}_2 = \boldsymbol{e}_3$, and $A\boldsymbol{e}_3 = \boldsymbol{e}_1$. (d) $A\boldsymbol{e}_1 = \boldsymbol{e}_2$, $A\boldsymbol{e}_2 = \boldsymbol{e}_3$, and $A\boldsymbol{e}_3 = \mathbf{0}$.

(e) Determine whether each matrix is invertible.

7. Find all 2×2 matrices that commute with $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Suggestion: Let B be a general matrix. Calculate AB and BA and set the corresponding entries equal.

8. Describe all 3×3 matrices A satisfying $\frac{1}{2}(A + A^{\mathsf{T}}) = I_3$.

9. Describe the set of all $\boldsymbol{b} = (b^1, b^2, b^3)$ in $\mathbf{R}^3$ for which the following linear system is consistent:

$$\begin{aligned} x^1 - x^2 - x^3 &= b^1, \\ 2x^1 + 2x^2 + x^3 &= b^2, \\ 4x^1 - x^3 &= b^3. \end{aligned}$$

10. Let $\boldsymbol{v}_1 = (3, 2)$ and $\boldsymbol{v}_2 = (1, 1)$.
(a) If $\boldsymbol{b} = (b^1, b^2)$ is in $\mathbf{R}^2$, show there exist scalars x^1 and x^2 such that $\boldsymbol{b} = x^1\boldsymbol{v}_1 + x^2\boldsymbol{v}_2$. Suggestion: Set up the preceding equation as a linear system, and solve.
(b) On a piece of graph paper, carefully sketch the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and the non-Cartesian grid they define. Use the grid and the parallelogram law (compare Figure 2.1, p. 27) to find scalars C^i_j such that $\boldsymbol{e}_j = \sum_i C^i_j \boldsymbol{v}_i$. Confirm that the answer agrees with the formula you found in part (a).
(c) How is the matrix $C = [C^i_j]$ related to the matrix $V = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$?

11. Let V be the set of functions of the form $f(x) = ae^x + be^{-x}$ for some real numbers a and b, and let $+$ and $\cdot$ denote ordinary addition and scalar multiplication of functions.
(a) Show that V is *closed under addition*, i.e., a sum of functions in V is in V.
(b) Show that V is *closed under scalar multiplication*, i.e., a scalar multiple of a function in V is in V.
(c) Consider the sets $\{e^x, e^{-x}\}$ and $\{\frac{1}{2}(e^x + e^{-x}), \frac{1}{2}(e^x - e^{-x})\}$. Make a sketch of these sets analogous to Example 2.16, p. 28.

12. Throughout, n is a positive integer, all matrices are of size $n \times n$, and all indices are between 1 and n.
(a) Show that $(\boldsymbol{e}^\ell_k)^i_j = \delta^i_k \delta^\ell_j$. Suggestion: Think about what each side means; feel free to make a verbal argument.
(b) Show that $(A\boldsymbol{e}^\ell_k)^i_j = \sum_{\alpha=1}^{n} A^i_\alpha (\boldsymbol{e}^\ell_k)^\alpha_j$, and simplify the right-hand side.
(c) Similarly calculate the (i, j)-entry of $\boldsymbol{e}^\ell_k A$.
(d) Suppose $[A, e^\ell_k] = 0$ for some k and ℓ. By taking $j = \ell$ and $i \neq k$, what can you deduce about the entries of A? What if $j = \ell$ and $i = k$?
(e) Suppose $[A, e^\ell_k] = 0$ for *all* k and ℓ. What can you deduce from the preceding part? Compare your conclusions with Question 1.10, p. 21.