College of the Holy Cross, Spring Semester, 2017
Math 242 (Professor Hwang)
Practice Problems for Midterm 1

The first midterm will be held in class on Friday, March 3. Coverage includes material up to the end of Chapter 4: sets, properties of the real numbers, inequalities, induction, intervals, upper and lower bounds, suprema and infima, sequences, convergence, subsequences, the Cauchy criterion, the Bolzano-Weierstrass theorem, infinite series and tests for (absolute) summability (the harmonic series, geometric series, comparison, ratio test, alternating series).

The questions below are, for the most part, more challenging than test questions (since you have access to notes and the text, and no one-hour time constraint), but it shouldn't be difficult to sketch solutions for them. The letters k and n stand for natural numbers, and x, y are real numbers.

1. In each part, negate the given statement, give a counterexample for the false statement, and prove the true statement.

   (a) For every real number x, $0 < x^2$.

   (b) There exists a real number x such that $x^2 = -1$.

   (c) For all real numbers x and y, if $0 < x < y$, then $0 < x^2 < y^2$.

2. In each part, k denotes an arbitrary positive integer. Find, with justification, the supremum and infimum of each of the following sets:

   (a) $A = \{\frac{(-1)^k}{k}\}$. (b) $A = \{(-1)^k(1 + \frac{1}{k})\}$. (c) $A = \{(-1)^k k\}$. (d) $A = \{k^{(-1)^k}\}$.

3. Let $A \subseteq \mathbf{R}$ be non-empty. Prove or (by giving a justified counterexample) disprove:

   (a) If A is bounded, then $\inf A < \sup A$.

   (b) If $-A = \{-a \mid a \in A\}$, then $\sup(-A) = -\inf A$.

   (c) If $\inf A < x < \sup A$, then $x \in A$

   (d) There exists an x in A such that $\sup A - \varepsilon < x$ for every $\varepsilon > 0$.

   (e) In each "false" part, characterize the sets for which the statement is true.

4. Let $b > 0$ be a real number, and consider the set $A = \{x \in \mathbf{R} : x^2 < b\}$. Prove:

   (a) A is non-empty, and bounded above by $1 + b$.

   (b) If $\beta = \sup A$, then $\beta^2 = b$.
   Hint: Show that if x is a real number such that $x^2 < b$, there exists $\varepsilon > 0$ such that $(x + \varepsilon)^2 < b$ (i.e., x is not an upper bound of A), while if x is positive and $x^2 > b$, there exists $\varepsilon > 0$ such that $(x - \varepsilon)^2 > b$ (i.e., x is not the least upper bound of A).

5. Explain the ε-N game in detail: What information is specified in advance? What constitutes "Player ε's turn"? What constitutes "Player N's turn"? How do we decide who wins the round? What does it mean (in terms of the game) to say that $(a_k) \to a_\infty$? How do we express the statement "$(a_k) \to a_\infty$" formally in analysis?

6. Let $b > 1$ be a real number. Define a real sequence (a_k) recursively by

$$a_0 = b, \quad a_{k+1} = \frac{1}{2}\left(a_k + \frac{b}{a_k}\right) = \frac{a_k^2 + b}{2a_k}.$$

   (Compare Example 4.30, p. 60.) Prove:

   (a) $a_k^2 > b$ for all k. Hint: Use induction to show that $a_k^2 - b > 0$.

   (b) (a_k) is strictly decreasing. Hint: Use induction to show $a_k - a_{k+1} > 0$.

   (c) There exists a real number a_∞ such that $(a_k) \to a_\infty$.

   (d) $a_\infty^2 = b$. Hint: $(a_{k+1}) \to a_\infty$.

7. Let (a_k) and (b_k) be sequences of real numbers. In each part, give a proof of the assertion (if True) or provide a counterexample (if False).

   (a) $(a_k) \to \infty$ implies (a_k) is non-decreasing.

   (b) If (a_k) is monotone, then (a_k) converges.

   (c) If $|a_k| \leq b_k$ and (b_k) converges, then (a_k) converges.

   (d) If $a_k \leq b_k$ and $(b_k) \to 0$, then (a_k) converges.

   (e) If $(a_k) \to 0$, then $(a_k b_k) \to 0$.

8. Let (a_k) and (b_k) be real sequences.

   (a) Assume (a_k) is bounded below. Prove that either (a_k) has a convergent subsequence, or has a subsequence that diverges to ∞.

   (b) Suppose $a_k \leq b_k$ for all k. Prove that if $(a_k) \to \infty$, then $(b_k) \to \infty$.

   (c) Suppose $a_k = k$, so that $(a_k) \to \infty$. Give an example of a sequence $(b_k) \to 0$ such that $b_k \neq 0$ for all k and the product sequence $(a_k b_k)$ exhibits the following behaviors: (i) converges to 0; (ii) converges to a positive real limit c; (iii) diverges to ∞; (iv) is bounded, but does not converge; (v) has no bounded subsequence, but does not diverge to ∞ or to $-\infty$; (vi) is unbounded, but has a subsequence converging to π and a subsequence converging to 0.

9. For each $k > 0$, let $a_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$.

   (a) Evaluate the partial sum $\displaystyle\sum_{k=1}^{n} a_k$.

   (b) Determine whether (a_k) is summable. If so, evaluate the infinite sum.

   (c) Determine whether $(\frac{1}{k^2})_{k=1}^{\infty}$ is summable. (Do not attempt to sum the series.)

10. Determine (with justification) whether each series is absolutely convergent, conditionally convergent, or divergent:

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$, (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$, (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k+1}$, (e) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{4^k}$.