College of the Holy Cross, Spring Semester, 2017
Math 242 (Professor Hwang)
Quiz 2 February 28, 2017

1. Let x and y be real numbers satisfying $2 \leq |x| \leq 3$ and $5 \leq |y| \leq 9$. Find the largest real number a and the smallest real number b such that $a \leq |x+y| \leq b$.

2. Let $x < 0$ and y denote real numbers satisfying $x < y$.

   (a) Is it **always** true that $x^2 < y^2$?

   (b) Is it **ever** true that $x^2 < y^2$?

   (c) Is it **always** true that $x^3 < y^3$?

3. Let x and y denote real numbers in $(-1, 1)$.

   (a) Show that $x + y < 1 + xy$. Hint: Consider $(1-x)(1-y)$.

   (b) Show that $\left|\dfrac{x+y}{1+xy}\right| < 1$.

   (In the language of Algebraic Structures, $x \oplus y = \dfrac{x+y}{1+xy}$ defines a binary operation on $(-1, 1)$. This operation turns out to be associative and commutative; it has an indentity element, and every element has an inverse.)

4. Let $(a_k)_{k=1}^{\infty}$ be the sequence defined by $a_k = \dfrac{2k}{k^2+1}$.

   (a) Find the first four terms, and the limit a_∞.

   (b) Find the smallest positive N such that if $k \geq N$, then $|a_k - a_\infty| < \frac{1}{50}$.

5. Recall that if $|r| < 1$, then $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$. Evaluate:

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{2 \cdot 5^k}{7^k}$; (b) $\displaystyle\sum_{k=0}^{\infty} \frac{-4 \cdot 5^k + (-2)^k}{7^k}$; (c) $\displaystyle\sum_{k=m}^{\infty} ar^k$.

6. Determine whether each series converges: (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1.0001^k}{k^{100}}$; (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{100}}{1.0001^k}$.

7. Let $A = (-5, 3) \cup (7, 10)$.

   (a) Prove that if $x \in A$, there exist y and z in A such that $y < x < z$.

   (b) Find the supremum and infimum of A, with justification. (Your justification should include precise definitions of sup and inf.)