College of the Holy Cross, Spring Semester, 2017
Math 242 (Professor Hwang)
Quiz 1 February 17, 2017

1. Let A be a set of real numbers.

   (a) Give a formal definition of the condition "A is bounded above".

   (b) State the completeness axiom for the real number system.

   (c) Give the formal definition of a supremum (least upper bound) of A. Phrase your answer in two forms, one the contrapositive of the other.

2. Suppose $A = \{\frac{1}{n} : n \geq 1\}$. Find the supremum and infimum of A with justification from the definitions.

3. Give examples of sequences of nested, non-empty open intervals ($I_{n+1} \subset I_n$ for all n) such that

   (a) $\bigcap_{n=1}^{\infty} I_n$ is empty. (b) $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

4. Let A and B be non-empty sets of real numbers, and assume $a \leq b$ for every a in A and every b in B. Prove that $\sup A \leq \inf B$:

   (a) Directly (from the definitions).

   (b) Contrapositively (assuming that if $\inf B < \sup A$, then there exists an a in A and a b in B such that $b < a$).

5. Construct a real sequence whose image is

   (a) The set of integers.

   (b) The set of rational numbers.
   Hint: Plot the points (p, q) with p and q integers and $q > 0$, then find a path that starts at $(p_0, q_0) = (0, 1)$ and visits each point exactly once. If (p_k, q_k) is the kth point visited, put $a_k = \frac{p_k}{q_k}$.

6. Let (a_k) be a real sequence that is bounded above, and define a new real sequence (α_n) by
$$\alpha_n = \sup\{a_k : n \leq k\}.$$

   (a) Prove that (α_k) is non-increasing.

   (b) Give an example of a non-constant sequence (a_k) for which (α_n) is constant.

   (c) Give an example of a sequence (a_k) for which (α_n) diverges.

7. Who was Paul Erdös? What did he mean when he asked, "How are your epsilons?"