College of the Holy Cross, Spring Semester, 2017
Math 242 (Professor Hwang)
Quiz 1 February 17, 2017

1. Let A be a set of real numbers.

   (a) Give a formal definition of the condition "A is bounded above".

   Solution There exists a real number U such that $x \leq U$ for all x in A.

   (b) State the completeness axiom for the real number system.

   Solution If $A \subseteq \mathbf{R}$ is non-empty and bounded above, then A has a real supremum.

   (c) Give the formal definition of a supremum (least upper bound) of A. Phrase your answer in two forms, one the contrapositive of the other.

   Solution Let $A \subseteq \mathbf{R}$ be a set. A real number β is a *supremum* of A if:

   (i) $x \leq \beta$ for all x in A;
   (ii) $\beta \leq U$ for every upper bound U of A.

   Contrapositively, (ii) can be replaced by
   (ii′) For every $\varepsilon > 0$, there exists an x in A such that $\beta - \varepsilon < x$.

2. Suppose $A = \{\frac{1}{n} : n \geq 1\}$. Find the supremum and infimum of A with justification from the definitions.

   Solution We have $\sup A = 1$ and $\inf A = 0$: If n is a natural number, then $\frac{1}{n} \leq 1$. Particularly, 1 is an upper bound of A, which implies $\sup A \leq 1$. Since $1 \in A$, every upper bound U of A satisfies $1 \leq U$; by (ii), $\sup A = 1$.

   For every natural number n, $0 < \frac{1}{n}$. That is, 0 is a lower bound of A, so $0 \leq \inf A$. If $\varepsilon > 0$, there exists a natural number n such that $\frac{1}{n} < \varepsilon$, so ε is not a lower bound of A. By (ii′), $\inf A = 0$.

3. Give examples of sequences of nested, non-empty open intervals ($I_{n+1} \subseteq I_n$ for all n) such that

   (a) $\bigcap_{n=1}^{\infty} I_n$ is empty. (b) $\bigcap_{n=1}^{\infty} I_n$ is non-empty.

   Solution You should have no trouble proving that

   $$\bigcap_{n=1}^{\infty}(0, \tfrac{1}{n}) = \varnothing, \qquad \bigcap_{n=1}^{\infty}(-\tfrac{1}{n}, \tfrac{1}{n}) = \{0\}, \qquad \bigcap_{n=1}^{\infty}(-\tfrac{n+1}{n}, \tfrac{n+1}{n}) = [-1, 1],$$

   etc.

4. Let A and B be non-empty sets of real numbers, and assume $a \leq b$ for every a in A and every b in B. Prove that $\sup A \leq \inf B$:

   (a) Directly (from the definitions).

   Solution By hypothesis, each b in B is an upper bound of A, so $\sup A \leq b$ for all b in B by the definition of a supremum. This in turn means $\sup A$ is a lower bound of B. The definition of an infimum immediately implies $\sup A \leq \inf B$.

(b) Contrapositively (assuming that if $\inf B < \sup A$, then there exists an a in A and a b in B such that $b < a$).

Solution If $\inf B < \sup A$, consider the real number $\gamma = \frac{1}{2}(\inf B + \sup A)$. Since $\inf B < \gamma < \sup A$, the contrapositive formulation of a sup or inf implies there exists a b in B such that $b < \gamma$, and there exists an a in A such that $\gamma < a$. Daisy-chaining, $b < \gamma < a$.

5. Construct a real sequence whose image is

(a) The set of integers.

(b) The set of rational numbers.
Hint: Plot the points (p, q) with p and q integers and $q > 0$, then find a path that starts at $(p_0, q_0) = (0, 1)$ and visits each point exactly once. If (p_k, q_k) is the kth point visited, put $a_k = \frac{p_k}{q_k}$.

Solution For example, define $a_k = \frac{1}{2}k$ if k is even, and $-\frac{1}{2}(k+1)$ if k is odd. It's straightforward to check that every integer is in the image.

A sequence enumerating the rationals is not easy to write down as a formula, but the hint should convince you that such a sequence exists.

6. Let (a_k) be a real sequence that is bounded above, and define a new real sequence (α_n) by

$$\alpha_n = \sup\{a_k : n \leq k\}.$$

(a) Prove that (α_k) is non-increasing.

Solution For each n, define $A_n = \{a_k : k \geq n\}$. Clearly $A_{n+1} \subseteq A_n$ for all n, so $\alpha_{n+1} = \sup A_{n+1} \leq \sup A_n = \alpha_n$ for all n.

(b) Give an example of a non-constant sequence (a_k) for which (α_n) is constant.

Solution If $a_k = -\frac{1}{k}$, then $\alpha_n = 0$ for all n.

(c) Give an example of a sequence (a_k) for which (α_n) diverges.

Solution If $\alpha_k = -k$, then $\alpha_n = -n$ for each n.

7. Who was Paul Erdös? What did he mean when he asked, "How are your epsilons?"

Solution Paul Erdös was an itinerant Hungarian mathematician who spent the last decades of his life living from a suitcase, traveling and collaborating, turning coffee into theorems. He referred to children as "epsilons", and would have been inquiring after one's offspring.