College of the Holy Cross
Math 135 (Calculus I)
Worksheet 6: Cosine and Sine

The *unit circle* in the (x, y)-plane has equation $x^2 + y^2 = 1$. Each point (x, y) on the unit circle lies on a ray through the origin making an *angle* with the positive x-axis.

The numerical value of the angle is the *arc length* from $(1, 0)$ to (x, y), measured counter-clockwise. Usually we let θ (the Greek letter "theta") stand for an angle.

Since the circumference of the unit circle is 2π by definition, an angle of 2π represents one full turn around the circle. Sometimes we let $[0, 2\pi]$ be the range of θ; other times it's more convenient to use $[-\pi, \pi]$. Neither is "more right" than the other.

1. (a) Show that $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ lie on the unit circle.

$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$ $\quad\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$ $\quad\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

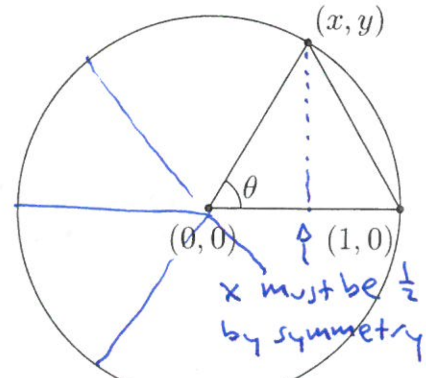
(b) Suppose the points $(0, 0)$, $(1, 0)$, and (x, y) are corners of an *equilateral triangle.*

(i) What are the numerical values of x and y? Hint: The triangle has a vertical axis of symmetry. $x = \frac{1}{2}$ $\therefore y = \frac{\sqrt{3}}{2}$

(ii) What is θ? Hint: How many equilateral triangles fit around the origin?

$\theta = \frac{1}{6}$ of a full turn $= \frac{2\pi}{6} = \frac{\pi}{3}$

or $\frac{360^\circ}{6} = 60^\circ$



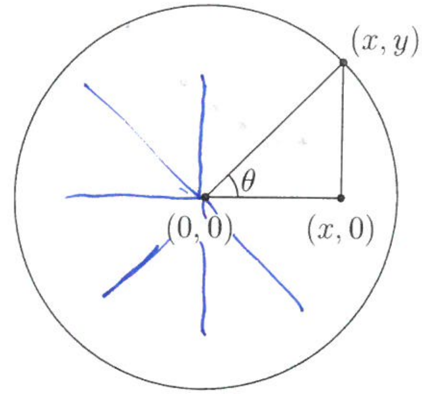
(c) Suppose the points $(0, 0)$, $(x, 0)$, and (x, y) are corners of a *right isosceles triangle.*

(i) What is the numerical value of x? Hint: Find y in terms of x. $x = y = \frac{\sqrt{2}}{2}$ (done in class)

(ii) What is θ? Hint: How many right isosceles triangles fit around the origin?

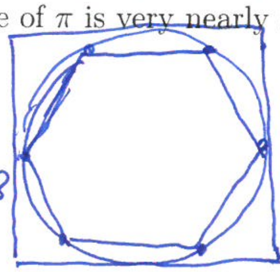
$\theta = \frac{1}{8}$ of a turn $= \frac{2\pi}{8} = \frac{\pi}{4}$ radians

or $\frac{360^\circ}{8} = 45^\circ$



(d) Use suitable inscribed and circumscribed polygons to show $6 < 2\pi < 8$, i.e., that $3 < \pi < 4$. (The actual value of π is very nearly $355/113 \approx 3.14159\ldots$.)

Inscribed hexagon has perimeter 6
circumscribed square has perimeter 8



2. If (x, y) is the point where the ray at angle θ cuts the unit circle, we call x the *cosine* of θ and y the *sine* of θ, and write

$$x = \cos\theta, \qquad y = \sin\theta.$$

(a) Use geometry to fill in the table entries. Use square roots, not approximate numerical values. Hint: You computed several values in the preceding question.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

(b) How are $\cos(-\theta)$ and $\cos\theta$ related? How are $\sin(-\theta)$ and $\sin\theta$ related?
Hint: When you measure angles clockwise instead of counter-clockwise, what effect is there on the horizontal (x) and vertical (y) position? $\cos(-\theta) = +\cos\theta$
$\sin(-\theta) = -\sin\theta$

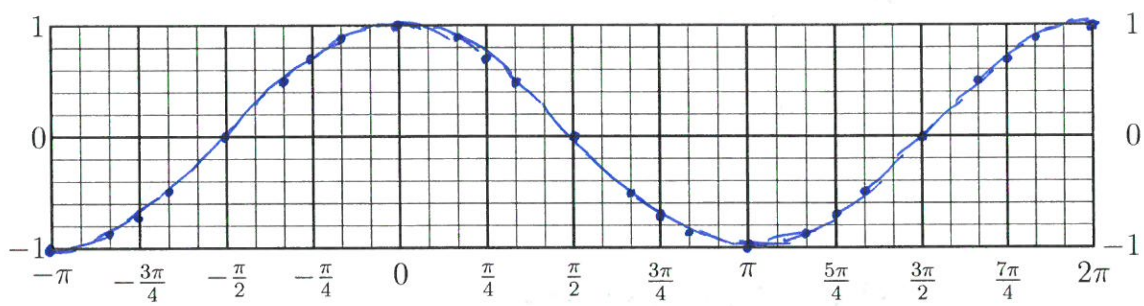
(c) Explain why, for all real θ, we have

$$-1 \le \cos\theta \le 1; \qquad -1 \le \sin\theta \le 1; \qquad \cos^2\theta + \sin^2\theta = 1.$$ $\rightarrow$ Pythagorean theorem!

The circle $x^2+y^2=1$ lies between the lines $x=1$, $x=-1$, and between $y=1$, $y=-1$.

3. Using the data of the previous question and the numerical approximations $\frac{\sqrt{2}}{2} \approx 0.7$ and $\frac{\sqrt{3}}{2} \approx 0.87$, carefully plot:

(a) The cosine function, $y = \cos\theta$, for $-\pi \le \theta \le 2\pi$:



(b) The sine function, $y = \sin\theta$, for $-\pi \le \theta \le 2\pi$: