College of the Holy Cross
Math 135 (Calculus I)
Worksheet 8A: Exponential Functions, I

1. Suppose someone agreed to pay you one dollar today, then two dollars tomorrow, then four dollars, and so on, doubling the amount each day.

(a) Give your best estimate of how many days it would be before the day's amount was: One hundred dollars; one million dollars; one trillion (a million million) dollars.

(b) Calculating by hand, start with 1 and double successively. Continue until you reach a number larger than *one million*. (More than enough spaces are provided.)

1	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ 2	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ 4	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ ________	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ ________	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ ________	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ ________	$\times 2 =$ ________	$\times 2 =$ ________
$\times 2 =$ ________	$\times 2 =$ ________	$\times 2 =$ ________

(c) About how many more doublings will it take before you reach one trillion? Why? Hint: You found a power of two just a bit larger than one million. Think of one million as a new unit of counting; about how many doublings until you reach one million million?

2. Products of the type $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ are difficult to read. Instead, to express a product containing nine factors of 2, we write 2^9, read "two to the ninth power", or "two to the ninth". Generally, if n is a positive integer, then 2^n means "multiply together n factors of 2". We call n the *exponent* or *power*.

(a) Express $2^2 \times 2^3$ as a power of 2. Hint: How many factors of 2 are there in the product? Can you check your answer by some independent method?

(b) Express $2^{54} \times 2^{17} \times 2^{128}$ as a power of 2.

(c) Express $(2^2)^3 = 2^2 \times 2^2 \times 2^2$ as a power of 2. Hint: How many factors of 2 are there in the product? Can you check your answer by some independent method?

(d) Suppose m and n are positive whole numbers. Explain why

$$2^m \times 2^n = 2^{m+n}, \qquad (2^m)^n = 2^{mn}.$$

(e) Express $2^5 \div 2^3$ as a power of 2. Hint: How many factors of 2 are there in the quotient? Can you check your answer by some independent method?

(f) Express $2^{128} \div 2^{54}$ as a power of 2.

(g) What formula summarizes the two preceding parts?

3. In the preceding question, we saw that

$$2^m \times 2^n = 2^{m+n}, \qquad 2^m \div 2^n = 2^{m-n}, \qquad (2^m)^n = 2^{mn}.$$

Our goal is to explore the properties of the *exponential function* $f(x) = 2^x$ that satisfies these properties for real numbers x (not just for positive whole numbers).

(a) Show that $2^{1/2} = \sqrt{2}$. Hint: What is $(2^{1/2})^2$?

(b) What is $2^{1/3}$? $2^{1/4}$? $2^{1/q}$ if q is a positive whole number?

(c) We can calculate $2^{3/2}$ in (at least) three ways:

$$2^{1+\frac{1}{2}} = 2 \times 2^{1/2}, \qquad 2^{2-\frac{1}{2}} = 2^2 \div 2^{1/2}, \qquad 2^{3\times\frac{1}{2}} = (2^3)^{1/2}.$$

Show that each gives the same value. Hint: Square each expression.

(d) We define $2^0 = 1$ and $2^{-n} = 1 \div 2^n = \dfrac{1}{2^n}$. Why?

4. In each part, $f(x) = 2^x$

(a) Show $f(0) = 1$. Hint: Just do it.

(b) Show $f(x+1) = 2f(x)$ for all real x.

(c) Show $f(x+\frac{1}{2}) = \sqrt{2}f(x) \approx 1.4f(x)$ for all x.

(d) Show $f(x-\frac{1}{2}) = \sqrt{\frac{1}{2}}f(x) \approx 0.7f(x)$ for all x.

(e) The preceding parts tell you the graph $y = 2^x$ passes through $(0,1)$, that the height scales by a factor of about 1.4 for each rightward step by $\frac{1}{2}$, and scales by a factor of about 0.7 for each leftward step by $\frac{1}{2}$.

Use these properties to plot several points, then carefully sketch the graph $y = 2^x$.