## College of the Holy Cross

Math 135 (Calculus I)

## Worksheet 6: Cosine and Sine

The unit circle in the $(x, y)$-plane has equation $x^{2}+y^{2}=1$. Each point $(x, y)$ on the unit circle lies on a ray through the origin making an angle with the positive $x$-axis.

The numerical value of the angle is the arc length from $(1,0)$ to $(x, y)$, measured counterclockwise. Usually we let $\theta$ (the Greek letter "theta") stand for an angle.

Since the circumference of the unit circle is $2 \pi$ by definition, an angle of $2 \pi$ represents one full turn around the circle. Sometimes we let $[0,2 \pi]$ be the range of $\theta$; other times it's more convenient to use $[-\pi, \pi]$. Neither is "more right" than the other.

1. (a) Show that $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ lie on the unit circle.
(b) Suppose the points $(0,0),(1,0)$, and $(x, y)$ are corners of an equilateral triangle.
(i) What are the numerical values of $x$ and $y$ ? Hint: The triangle has a vertical axis of symmetry.
(ii) What is $\theta$ ? Hint: How many equilateral triangles fit around the origin?

(c) Suppose the points $(0,0),(x, 0)$, and $(x, y)$ are corners of a right isosceles triangle.
(i) What is the numerical value of $x$ ? Hint: Find $y$ in terms of $x$.
(ii) What is $\theta$ ? Hint: How many right isosceles triangles fit around the origin?

(d) Use suitable inscribed and circumscribed polygons to show $6<2 \pi<8$, i.e., that $3<\pi<4$. (The actual value of $\pi$ is very nearly $355 / 113 \approx 3.14159 \ldots$.)
2. If $(x, y)$ is the point where the ray at angle $\theta$ cuts the unit circle, we call $x$ the cosine of $\theta$ and $y$ the sine of $\theta$, and write

$$
x=\cos \theta, \quad y=\sin \theta .
$$

(a) Use geometry to fill in the table entries. Use square roots, not approximate numerical values. Hint: You computed several values in the preceding question.

| $\theta$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\cos \theta$ |  |  |  |  |  |  |  |  |  |
| $\sin \theta$ |  |  |  |  |  |  |  |  |  |

(b) How are $\cos (-\theta)$ and $\cos \theta$ related? How are $\sin (-\theta)$ and $\sin \theta$ related?

Hint: When you measure angles clockwise instead of counter-clockwise, what effect is there on the horizontal $(x)$ and vertical $(y)$ position?
(c) Explain why, for all real $\theta$, we have

$$
-1 \leq \cos \theta \leq 1 ; \quad-1 \leq \sin \theta \leq 1 ; \quad \cos ^{2} \theta+\sin ^{2} \theta=1
$$

3. Using the data of the previous question and the numerical approximations $\frac{\sqrt{2}}{2} \approx 0.7$ and $\frac{\sqrt{3}}{2} \approx 0.87$, carefully plot:
(a) The cosine function, $y=\cos \theta$, for $-\pi \leq \theta \leq 2 \pi$ :

(b) The sine function, $y=\sin \theta$, for $-\pi \leq \theta \leq 2 \pi$ :

