College of the Holy Cross
Math 135 (Calculus I)
Worksheet 5: Rational Functions

1. On a single set of axes, sketch the graphs $y = \dfrac{1}{1+x^2}$, $y = \dfrac{x}{1+x^2}$, $y = \dfrac{x^2}{1+x^2}$.

2. On a single set of axes, sketch the graphs $y = \dfrac{1}{1-x^2}$, $y = \dfrac{x}{1-x^2}$, $y = \dfrac{x^2}{1-x^2}$.

3. The table below gives values of $u(t) = \dfrac{2t}{t^2+1}$ and $v(t) = \dfrac{t^2-1}{t^2+1}$ for $0 \leq t \leq 3$.

t	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$u(t)$	0.00	0.20	0.38	0.55	0.69	0.80	0.88	0.94	0.98	0.99
$v(t)$	-1.00	-0.98	-0.92	-0.83	-0.72	-0.60	-0.47	-0.34	-0.22	-0.10
t	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90
$u(t)$	1.00	1.00	0.98	0.97	0.95	0.92	0.90	0.87	0.85	0.82
$v(t)$	0.00	0.10	0.18	0.26	0.32	0.38	0.44	0.49	0.53	0.57
t	2.00	2.10	2.20	2.30	2.40	2.50	2.60	2.70	2.80	2.90
$u(t)$	0.80	0.78	0.75	0.73	0.71	0.69	0.67	0.65	0.63	0.62
$v(t)$	0.60	0.63	0.66	0.68	0.70	0.72	0.74	0.76	0.77	0.79

(a) Carefully sketch the graphs $y = u(t)$ and $y = v(t)$ in the (t, y)-plane for $0 \leq t \leq 3$. Show that $u(-t) = -u(t)$, and sketch the left half of the graph $y = u(t)$.

Similarly, show $v(-t) = v(t)$ for all t, and sketch the left half of the graph $y = v(t)$.

(b) Carefully plot the points $\big(u(t), v(t)\big)$ in the (u, v)-plane.

Use algebra to show that $u(1/t) = u(t)$ for all $t \neq 0$, and that $v(1/t) = -v(t)$ for $t \neq 0$. Hint: Multiply $u(1/t)$ by t^2/t^2. Proceed similarly for $v(1/t)$.

(c) Plot the points $(0, 1)$, $\big(u(1/2), v(1/2)\big)$, and $(1/2, 0)$. Do you notice anything? What about the points $(0, 1)$, $\big(u(2), v(2)\big)$, and $(2, 0)$?

4. In each part, let

$$f(x) = \frac{10x^2 - 30x + 7}{x^3 + 1}, \qquad g(x) = \frac{10x^3 - 30x + 7}{x^3 + 1}, \qquad h(x) = \frac{0.1x^4 - 30x + 7}{x^3 + 1}.$$

(a) Does each function have any vertical asymptotes? Horizontal asymptotes?

(b) For which positive real x do we have $f(x) = g(x)$? $f(x) = h(x)$? $g(x) = h(x)$?

(c) Put the three functions in order (smallest to largest) if $0 < x < 1$; $1 < x < 10$; $10 < x < 100$; $100 < x$.

5. In each part, let $f(x) = 5x^2 + 3x - 100$, $g(x) = 4.999x^2$, and $h(x) = 5.001x^2$.

(a) Show that $g(x) \leq 5x^2 \leq h(x)$ for all real x.

(b) Is it true that $g(x) \leq f(x) \leq h(x)$ for all real x? Explain.

(c) Show that $g(x) \leq f(x) \leq h(x)$ for sufficiently large $|x|$.