College of the Holy Cross
Math 135 (Calculus I)
Supplement 8: Counting Solutions of Equations

If f is continuous on the interval $[a, b]$ and increasing, then for every y with $f(a) \leq y \leq f(b)$, there is *exactly one* x with $a \leq x \leq b$ and $y = f(x)$.

Similarly, if f is continuous on the interval $[a, b]$ and decreasing, then for every y with $f(b) \leq y \leq f(a)$, there is *exactly one* x with $a \leq x \leq b$ and $y = f(x)$.

Example 1. Suppose $f(x) = \dfrac{2x}{x^2+1}$, so that $f'(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

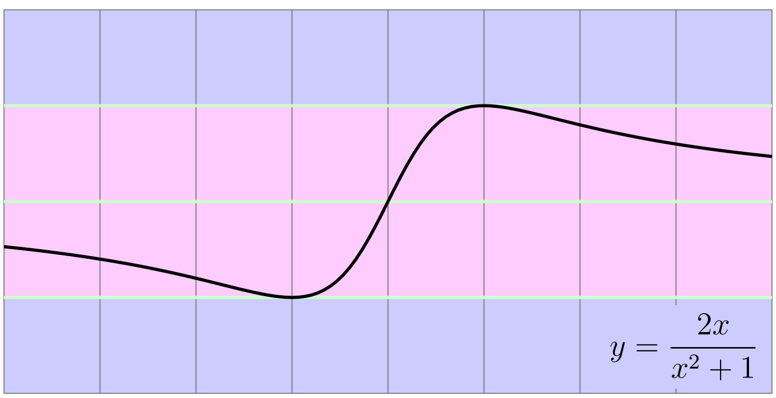
If $|y| > 1$, the equation $y = f(x)$ has no real solutions.

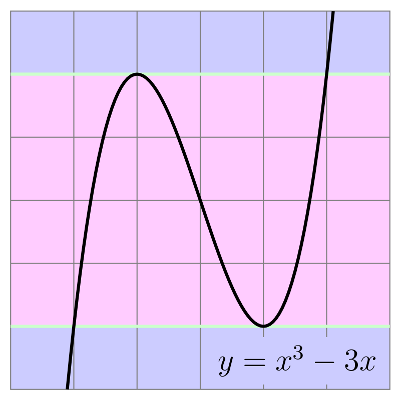
If $y = -1$, 0, or 1, the equation $y = f(x)$ has exactly one solution.

If $0 < |y| < 1$, the equation $y = f(x)$ has precisely two real solutions.

To see why, note that f has critical points at $x = 1$ and $x = -1$; f is increasing on $[-1, 1]$ and decreasing on $(-\infty, -1]$ and $[1, \infty)$, and finally that $f(x) \to 0$ as $x \to \pm\infty$.

We deduce that if $x \leq -1$, then $-1 = f(-1) \leq f(x) < 0$, and each value in this range is hit exactly once; if $-1 \leq x \leq 1$, then $-1 = f(-1) \leq f(x) \leq f(1) = 1$, and every value in this range is hit exactly once; if $1 \leq x$, then $0 < f(x) \leq f(1) = 1$, and every value in this range is hit exactly once.





Example 2. Suppose $f(x) = x^3 - 3x$, so that $f'(x) = 3(x^2 - 1)$.

If $|y| > 2$, the equation $y = f(x)$ has exactly one real solution.

If $y = -2$ or $y = 2$, the equation $y = f(x)$ has exactly two real solutions.

If $-2 < y < 2$, the equation $y = f(x)$ has precisely three real solutions.

To see why, note that f has critical points at $x = 1$ (where $y = f(1) = -2$) and $x = -1$ (where $y = f(-1) = 2$), f is increasing on $(-\infty, -1]$ and $[1, \infty)$ and decreasing on $[-1, 1]$, and $f(x) \to -\infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$.

We deduce that if $x \leq -1$, then $-\infty < f(x) \leq f(-1) = 2$ and every value in this range is hit exactly once; if $-1 \leq x \leq 1$, then $-2 = f(1) \leq f(x) \leq f(-1) = 2$ and every value in this range is hit exactly once; if $1 \leq x$, then $f(1) = -2 \leq f(x) < \infty$, and every value in this range is hit exactly once.