## Supplement 7: Derivatives and the Shape of a Graph

The first and second derivatives of a function tell us useful information about the shape of the graph. To be precise, but to avoid restating technical conditions, we'll assume in these notes that $f$ is continuous in some interval $[a, b]$ and $f^{\prime \prime}$ exists in the open interval $(a, b)$. (We also say $f$ is "twice differentiable".)

Definition 1. We say $f$ is increasing in $[a, b]$ if whenever $a \leq s<t \leq b$, we have $f(s)<f(t)$.
We say $f$ is non-decreasing in $[a, b]$ if whenever $a \leq s<t \leq b$, we have $f(s) \leq f(t)$.
Remark 2. An increasing function can be applied to an inequality to obtain a new inequality pointing in the same direction. For instance, the squaring function $f(x)=x^{2}$ is increasing on $[0, \infty)$, so if $2<\sqrt{s}<\sqrt{5}$, then $4<s<5$.

Example 3. In the interval $(-\infty, \infty)$ : The function $f(x)=x^{3}$ is increasing.
The signum function $\operatorname{sgn}(x)=x /|x|$ for $x \neq 0$, and $\operatorname{sgn}(0)=0$, is non-decreasing but not increasing.

The floor and ceiling functions are non-decreasing but not increasing.
A composition of increasing functions is increasing. (Why?) A composition of nondecreasing functions is non-decreasing. For instance, $f(x)=\left\lfloor x^{2}\right\rfloor$ is non-decreasing.

Activity 1. Give the definitions of decreasing and non-increasing. Give an example of a function that is decreasing, and of a function that is non-increasing but not decreasing.
Definition 4. We say $f$ is convex (or concave up) if whenever $a \leq s<u<t \leq b$, the point $(u, f(u))$ lies below the secant line through $(s, f(s))$ and $(t, f(t))$. Algebraically,

$$
f(u)<f(s)+\frac{f(t)-f(s)}{t-s}(u-s) .
$$



Activity 2. Give the definition of concave (down), both geometrically and as an algebraic condition, and illustrate with a sketch.

The main qualitative connection between first and second derivatives and the shape of a graph is summarized as follows:

Theorem 5. If $f^{\prime}>0$ on $(a, b)$, then $f$ is increasing in $[a, b]$.
If $f^{\prime}<0$ on $(a, b)$, then $f$ is decreasing in $[a, b]$.
If $f^{\prime \prime}>0$ on $(a, b)$, then $f$ is convex in $[a, b]$.
If $f^{\prime \prime}<0$ on $(a, b)$, then $f$ is concave in $[a, b]$.
The key technical property behind these results is the Mean Value Theorem, or MVT.
Theorem 6 (Mean Value Theorem). If $f$ is continuous on $[a, b]$, and is differentiable in $(a, b)$, then there is an least one $c$ with $a<c<b$ and

$$
f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}
$$

Remark 7. In words, if $f$ is differentiable on an interval $[a, b]$, then at some point in the open interval $(a, b)$, the instantaneous rate of change of $f$ must equal the average rate of change of $f$ over $[a, b]$.

Physically, if you drive 60 miles in some one-hour period, your speed has to be 60 mph at least once. (Mathematically, your speed could also be a constant 60 mph . Reality usually lies somewhere in between.)

Remark 8. A key feature of the MVT is, if $f$ satisfies the hypotheses of the MVT in some interval $[a, b]$, then $f$ also satisfies the hypotheses in every interval contained in $[a, b]$.

Reason for Theorem 5. Suppose $f^{\prime}(x)>0$ for all $x$ with $a<x<b$. If $a \leq s<t \leq b$, we may apply the MVT to $f$ on $[s, t]$, deducing there is a $c$ with $s<c<t$ such that

$$
f^{\prime}(c)=\frac{f(t)-f(s)}{t-s}, \quad \text { or } \quad f(t)-f(s)=f^{\prime}(c)(t-s)
$$

But $t-s$ and $f^{\prime}(c)$ are positive, so their product is positive; that is, $f(s)<f(t)$.
The second part of Theorem 5 is shown similarly. For the third and fourth, we use the same ideas but the details are more elaborate. If you're curious, see me in office hours!
Example 9. The converse of Theorem 5 is not true: If $f(x)=x^{3}$, then $f^{\prime}(x)=3 x^{2}=0$ at $x=0$. Despite this, $f$ is increasing!

We can use Theorem 5 to graph functions in a "holistic" way. (Plotting many points is the "reductionistic" way.)

Example 10. Let $f(x)=\left(x^{2}-4\right)^{2}$. Find the intervals where $f$ is increasing or decreasing, and convex or concave, and use these to sketch the graph.

We first calculate the first and second derivatives. Expanding, $f(x)=x^{4}-8 x^{2}+16$. By differentiation rules, we get

$$
\begin{aligned}
f^{\prime}(x) & =4 x^{3}-16 x=4 x\left(x^{2}-4\right)=4 x(x-2)(x+2) \\
f^{\prime \prime}(x) & =12 x^{2}-16=12\left(x^{2}-4 / 3\right)=12(x-2 / \sqrt{3})(x+2 / \sqrt{3})
\end{aligned}
$$

The sign of $f^{\prime}$ can change only where $f^{\prime}(x)=0$, i.e., at $x=-2, x=0$, or $x=2$. We conclude that $f^{\prime}$ has constant sign on the intervals $(-\infty,-2),(-2,0),(0,2)$, and $(2, \infty)$.

We can determine the sign on each interval by plugging in a "test point". For instance $f^{\prime}(1)=4(-1)(3)<0$, so $f^{\prime}(x)<0$ if $0<x<2$. The signs of $f^{\prime}$ may be found similarly for the other intervals.

Similarly, the sign of $f^{\prime \prime}$ can change only where $f^{\prime \prime}(x)=0$, i.e., at $x= \pm 2 / \sqrt{3}$. Since $f^{\prime \prime}(0)<0$, we deduce that $f^{\prime \prime}(x)<0$ for $-2 / \sqrt{3}<x<2 / \sqrt{3}$.

We may summarize our findings with sign diagrams:


Now we're ready to sketch. For each $x$ where $f^{\prime}(x)=0$ or $f^{\prime \prime}(x)=0$, we calculate a $y$ value and plot that point. Then we connect the dots, making sure the function is increasing or decreasing, and is convex or concave, as determined by the sign diagrams.

