College of the Holy Cross
Math 135 (Calculus I)
Supplement 6: Approximate Linearity

Suppose f is a differentiable function, let a be a point of the domain of f, and define $\ell(x) = f(a)+f'(a)(x-a)$, a function whose graph is the tangent line to $y = f(x)$ at $\bigl(a, f(a)\bigr)$.

The difference $f(x)-\ell(x)$ measures "how good an approximation" $\ell(x)$ is to $f(x)$. Write $E(x) = f(x) - \ell(x)$ (with E for "error"), so that

$$f(x) = f(a) + f'(a)(x - a) + E(x).$$

Since f is differentiable at a, we have

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Subtracting $f'(a)$ from both sides gives

$$0 = \lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = \lim_{x\to a} \frac{f(x) - \ell(x)}{x - a} = \lim_{x\to a} \frac{E(x)}{x - a}.$$

This equation says the function ℓ is so good an approximation to f near a that the error $E(x) = f(x) - \ell(x)$ is small *even compared to* $x - a$.

Conversely, if f is a function, and if m is a real number such that

$$0 = \lim_{x\to a} \frac{f(x) - f(a) - m(x - a)}{x - a},$$

then f is differentiable at a, and $f'(a) = m$. In words, a function f is differentiable at a if and only if f is "approximately linear" at a.

We can use approximate linearity to prove the product rule differently than the book does: Assume f and g are differentiable at a, so that

$$f(x) = f(a) + f'(a)(x - a) + E_f(x), \qquad g(x) = g(a) + g'(a)(x - a) + E_g(x).$$

Multiplying these and gathering like terms,

$$f(x)g(x) = f(a)g(a) + \bigl[f(a)g'(a) + f'(a)g(a)\bigr](x - a) + E(x),$$

where $E(x)/(x - a) \to 0$ as $x \to a$. The coefficient of $(x - a)$, namely $f(a)g'(a) + f'(a)g(a)$, must be the derivative of fg at a.

We can now prove the quotient rule in two stages. First, if g is differentiable at a and if $g(a) \neq 0$, then

$$\lim_{x\to a} \frac{\dfrac{1}{g(x)} - \dfrac{1}{g(a)}}{x - a} = \lim_{x\to a} \frac{\dfrac{g(a) - g(x)}{g(a)g(x)}}{x - a} = \lim_{x\to a} -\frac{g(x) - g(a)}{x - a} \cdot \frac{1}{g(a)g(x)} = -\frac{g'(a)}{g(a)^2}.$$

That is, $1/g$ is differentiable at a, and $(1/g)'(a) = -g'(a)/g(a)^2$. The product rule applied to $f/g = f \cdot (1/g)$ now gives

$$\left(\frac{f}{g}\right)'(a) = f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\frac{1}{g(a)} = -\frac{f(a)g'(a)}{g(a)^2} + \frac{f'(a)}{g(a)} = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}.$$