College of the Holy Cross
Math 135 (Calculus I)
Supplement 5: Notes on Mathematical Grammar
Points, Equations, Functions, Graphs, and Inequalities

Points

When we denote a point in the Cartesian plane by (x, y), we presume that an origin O and two coordinate axes X and Y are given. The ordered pair (x, y) is an "address", with x the east-west location and y the north-south location. The important conceptual point is, *every ordered pair of real numbers corresponds to precisely one location in the Cartesian plane, and every location arises this way.*

Remark 1. There is nothing "permanent" or "universal" about the notation (x, y). We could denote an arbitrary point in the OXY-plane by (u, v), or (x_0, y_0), or (a, b), for instance. What determines the meaning of a mathematical expression is not the letters in it, but the stated roles those letters play.

Equations

Example 2. Let a, b, and c be numbers. The equation $y = ax^2 + bx + c$ may be viewed as:

1. As a *constraint* or *condition* on points (x, y) in the plane. The set of (x, y) satisfying this equation is a parabola (if $a \neq 0$) or a line (if $a = 0$).

2. As defining a *function* f, writing the "dependent variable" y as a function of the "independent variable" x.

Remark 3. Note carefully that neither interpretation "knows anything about" the notation (x, y). The equation $v = au^2 + bu + c$ defines *exactly the same constraint* if (u, v) denotes Cartesian coordinates, and defines *exactly the same function* f.

Variables are just names we assign to quantities for convenience, e.g., so we can say "x" instead of "the horizontal position of a general point on the graph" or some such.

Remark 4. Here, we cannot use (a, b) as Cartesian coordinates: The letters a, b, and c have *pre-assigned meanings* as coefficients of a quadratic. If we write $b = a \cdot a^2 + b \cdot a + c$,[1] we have used a to denote two distinct things (the coefficient of a quadratic and a Cartesian coordinate), and similarly for b. This is a recipe for trouble.

Functions

Functions and sets of points in the plane exist and have mathematical properties even when we have not given them names. This matters to us: Calculus is the study of functions—of mathematical relationships that do not depend on the names we have given to variables.

Example 5. Suppose f and g are functions, and put $y = f(x)$ and $z = g(x)$.

When we write $(f + g)' = f' + g'$ (Newton notation), we are closer to mathematical truth than when we write $\frac{d(y+z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx}$ (Leibniz notation). Newton notation explicitly denotes *functional relationships*. Leibniz notation leaves functional relationships implicit: We have only written down their "shadows", the respective inputs and outputs, *whose names have nothing to do with the property being expressed*!

For better or worse, Leibniz notation is Very Convenient in practical applications.

[1] Got it?

Graphs

Suppose f is a function defined on some interval $[a, b]$. That is, $f(x)$ is defined for $a \leq x \leq b$, and $f(u)$ is defined for $a \leq u \leq b$, and $f(3)$ is defined for $a \leq 3 \leq b$.[2]

The equation $y = f(x)$ is a *logical condition* on points (x, y): For each (x, y), the equation $y = f(x)$ is either "true" or "false". The points (x, y) with $y = f(x)$ and $a \leq x \leq b$, i.e., the points $\big(x, f(x)\big)$ with $a \leq x \leq b$, constitute the *graph of* f.

The graph of f is also the set of points $\big(u, f(u)\big)$ with $a \leq u \leq b$, or the set of points $\big(x_0, f(x_0)\big)$ with $a \leq x_0 \leq b$. (Using "x_0" to denote an arbitrary real number in $[a, b]$ is usually a recipe for trouble, though not as bad as using "3".)

Inequalities

Suppose f is a function with domain $[a, b]$. The inequality $y < f(x)$, for $a \leq x \leq b$, is again a *logical condition* on points (x, y). What does this inequality mean?

To answer, we have to think "outside the $y = f(x)$ box": The symbols x and y in "$y = f(x)$" are logically incompatible with the same symbols in "$y < f(x)$"! If we use (x, y) in both, we'll be in as much trouble as if we used (a, b) as Cartesian coordinates when studying the quadratic with coefficients a, b, and c.

Instead, let's view the graph of f as the set of points $\big(x, f(x)\big)$ with $a \leq x \leq b$, so we don't "waste" the letter y redundantly.

Now: For which (x, y) with $a \leq x \leq b$ is $y < f(x)$ true?

Let's think of x as fixed. The set of points (x, y) is the vertical line with horizontal position x. The particular point $\big(x, f(x)\big)$ lies on the graph. A point (x, y) with $y < f(x)$ lies "below" or "south of" the graph. A point (x, y) with $f(x) < y$ lies "above" or "north of" the graph. The single point (x, y) with $y = f(x)$ lies on the graph.

Group Worksheet 2, and Beyond

On the second group worksheet, and in realistic application of calculus beyond the course, the preceding issues become crucial to understand and to be able to use yourself.

In Question 1, we have the parabola $y = x^2$, i.e., the graph of the squaring function f. The tangent line to the graph of f at a point (a, a^2) has equation $y - f(a) = f'(a)(x - a)$, or $y - a^2 = 2a(x - a)$. Rearranging gives the condition

$$y - 2ax + a^2 = 0.$$

This single equation may be viewed in two useful ways:

1. With a fixed, this condition on a point (x, y) is true precisely when (x, y) lies on the line tangent to the parabola at (a, a^2).

2. With (x, y) fixed, this condition on a real number a is true precisely when the line tangent to the parabola at (a, a^2) passes through (x, y).

We *found* this equation in Question 1(a) using the first interpretation, but we *employ* this equation in Question 1(b) using the second interpretation! Keep these remarks in mind as you read the worksheet solutions.

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[2]Heh. The symbol "3" has such a universally-recognized meaning as a specific integer/real number that to use 3 as a variable is also a recipe for trouble. It *is* funny, though.