College of the Holy Cross
Math 135 (Calculus I)
Supplement 4: Limits of Trig Functions (October 5)

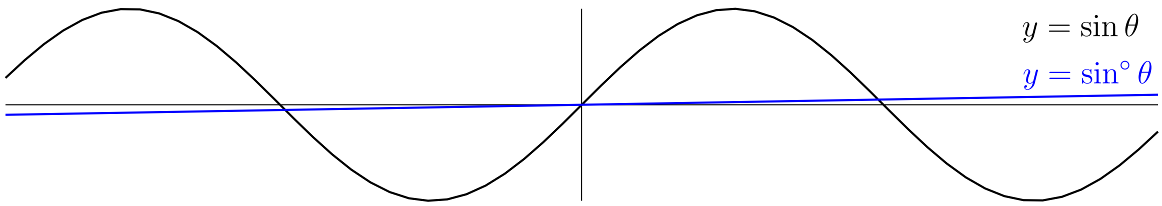
Sine in Degrees The *sine function in degrees* is the sinusoid whose period is 360, the number of degrees in one full turn of a circle. In terms of the "ordinary" sine function (in radians),

$$\sin^\circ \theta = \sin(\tfrac{\pi}{180}\theta).$$

The multiplier $\frac{\pi}{180}$ is the number of radians in one degree of angle, and converts radians to degrees.

By calculations from class,

$$\lim_{\theta\to 0} \frac{\sin^\circ \theta}{\theta} = \lim_{\theta\to 0} \frac{\sin(\frac{\pi}{180}\theta)}{\theta} = \frac{\pi}{180}.$$



The tangent line to the graph $y = \sin\theta$ at $(0, 0)$ has equation $y = \theta$. Again, this is why we use radians to measure angle in calculus.

The tangent line to the graph $y = \sin^\circ \theta$ at $(0, 0)$ has less pleasant equation $y = \frac{\pi}{180}\theta$. To graph one full oscillation of $\sin^\circ$ with the vertical scale above would require a plot 180 inches (15 feet) wide.

Polynomial Approximations Cosine and sine turn out to have approximations

$$\begin{aligned}\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \\ &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \dots,\end{aligned} \qquad \begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \\ &= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \frac{\theta^7}{5040} + \dots.\end{aligned}$$

The pattern of coefficients continues, and in a well-defined mathematical sense, the equalities are exact when "the number of terms is infinite". You can see shadows of these "series expansions" in three formulas we deduced in class today:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \qquad \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \tfrac{1}{2}, \qquad \sin'\theta = \cos\theta.$$

(For the first two, rearrange the series and set $\theta = 0$. For the third, you'll need a formula we've implicitly deduced, that the derivative of $f(\theta) = \theta^n$ is $f'(\theta) = n\theta^{n-1}$.)

When we estimated that $\cos(0.01) \approx 1 - 0.5(0.01)^2$, the series representation of $\cos\theta$ shows that the error of this estimate is smaller than $\frac{1}{24}(0.01)^4 < 0.5 \times 10^{-9}$. This explains the claim made in class that this estimate is correct to nine decimal places.