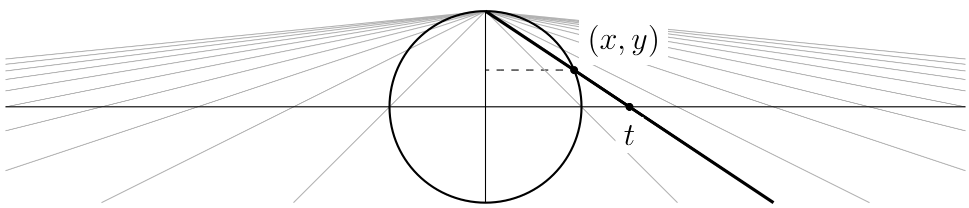
College of the Holy Cross
Math 135 (Calculus I)
Supplement 3: Stereographic Projection and Pythagorean Triples (October 1)

Stereographic Projection Let $(x, y) \neq (0, 1)$ be a point on the unit circle $x^2 + y^2 = 1$, and let $(t, 0)$ be the point where the ray from $(0, 1)$ to (x, y) crosses the x-axis. The triangle with vertices $(0, 1)$, (x, y), and $(0, y)$ is similar to the triangle with vertices $(0, 1)$, $(t, 0)$, and $(0, 0)$. It follows that the width-to-height ratios are the same: $t = x/(1-y)$, or $x = t(1-y)$.



To express (x, y) in terms of t, plug $x = t(1 - y)$ into the equation of the unit circle:

$$1 = x^2 + y^2 = t^2(1 - y)^2 + y^2,$$

or $1 - y^2 = t^2(1 - y)^2$. Since $y \neq 1$, we may factor the difference of squares on the left and cancel $1 - y$, giving $1 + y = t^2(1 - y) = t^2 - t^2 y$. Moving the ys to the left and the non-ys to the right, $y(t^2 + 1) = t^2 - 1$, or $y = (t^2 - 1)/(t^2 + 1)$. Finally,

$$x = t(1 - y) = t\left[1 - \frac{t^2 - 1}{t^2 + 1}\right] = t\,\frac{(t^2 + 1) - (t^2 - 1)}{t^2 + 1} = \frac{2t}{t^2 + 1}.$$

Thus, as claimed in class, $(x, y) = \left(\dfrac{2t}{t^2 + 1}, \dfrac{t^2 - 1}{t^2 + 1}\right)$. As $t \to \infty$ or $t \to -\infty$, $(x, y) \to (0, 1)$.

Pythagorean Triples If $t = p/q$ is a rational number (ratio of integers), then

$$(x, y) = \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right) = \left(\frac{2p/q}{(p/q)^2 + 1}, \frac{(p/q)^2 - 1}{(p/q)^2 + 1}\right) = \left(\frac{2pq}{p^2 + q^2}, \frac{p^2 - q^2}{p^2 + q^2}\right).$$

Conversely, if (x, y) is a point on the unit circle and x, y are both rational, then $t = x/(1-y)$ is rational. That is, our description of the unit circle gives a "one-to-one correspondence" between rational numbers and rational points on the unit circle other than $(0, 1)$.

If (A, B, C) is a Pythagorean triple, i.e., a triple of integers satisfying $A^2 + B^2 = C^2$ (and with $C \neq 0$), then $(x, y) = (A/C, B/C)$ is a rational point on the unit circle.

Conversely, if $(x, y) = (A/C, B/C)$ is a rational point on the circle, then (A, B, C) is a Pythagorean triple, and there exist integers p and q such that

$$A = 2pq, \qquad B = p^2 - q^2, \qquad C = p^2 + q^2.$$

If we assume $0 < q < p$, then A and B are positive. We may then read off as many distinct Pythagorean triples as we like, e.g.,

q, p	$1, 2$	$1, 3$	$2, 3$	$1, 4$	$3, 4$	$2, 5$	$1, 6$
A	4	6	12	8	24	20	12
B	3	8	5	15	7	21	35
C	5	10	13	17	25	29	37