College of the Holy Cross
Math 135 (Calculus I)
Supplement 2: Pascal's Triangle and the Binomial Theorem (September 26)

Computing $(a+h)^2 = (a+h)(a+h) = a^2 + 2ah + h^2$ is easy enough, but higher powers quickly become onerous. The insight leading to Pascal's triangle can be expressed as a "recursive" application of the distributive law. Since $(a+h)^3 = (a+h)^2(a+h)$, we have

$$\begin{aligned}
(a+h)^3 &= (a^2 + 2ah + h^2)(a+h) \\
&= (a^2 + 2ah + h^2)a + (a^2 + 2ah + h^2)h \\
&= \ a^3 + 2a^2h + \ \ ah^2 \\
&\qquad + \ \ a^2h + 2ah^2 + h^3 \\
&= a^3 + 3a^2h + 3ah^2 + h^3.
\end{aligned}$$

Similarly, we can use this cube formula to extend to fourth powers:

$$\begin{aligned}
(a+h)^4 &= (a^3 + 3a^2h + 3ah^2 + h^3)(a+h) \\
&= (a^3 + 3a^2h + 3ah^2 + h^3)a + (a^3 + 3a^2h + 3ah^2 + h^3)h \\
&= \ a^4 + 3a^3h + 3a^2h^2 + \ \ ah^3 \\
&\qquad + \ \ a^3h + 3a^2h^2 + 3ah^3 + h^4 \\
&= a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4.
\end{aligned}$$

If you want to be sure you understand, write out the calculations for the fifth power, convincing yourself that the coefficients arise by "duplicating" the sequence $1, 4, 6, 4, 1$, shifting one copy by one space, and adding to get $1, 5, 10, 10, 5, 1$.

Since the issue came up in class: To expand $(3x-2y)^4$, it would be easiest (given what we now know) to introduce $a = 3x$ and $h = -2y$, so that $3x - 2y = a + h$. Use the preceding formula to expand, and replace a by $3x$ and h by $-2y$:

$$\begin{aligned}
(3x-2y)^4 &= (3x)^4 + 4(3x)^3(-2y) + 6(3x)^2(-2y)^2 + 4(3x)(-2y)^3 + (-2y)^4 \\
&= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.
\end{aligned}$$

We won't need this in calculus, but in probability you very well might.

Finally, for those of you who like formulas, the coefficient of $a^{n-k}h^k$ in $(a+h)^n$ is the *binomial coefficient* $\binom{n}{k}$, read "n choose k", and equal to

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}.$$

(If $n > 0$ is an integer, the *factorial* $n!$ is the product of the integers from 1 to n. We define $0! = 1$.) For instance, the coefficients in the fourth power are

$$\binom{4}{0} = \binom{4}{4} = \frac{4!}{0!\,4!} = 1, \qquad \binom{4}{1} = \binom{4}{3} = \frac{4!}{1!\,3!} = 4, \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6,$$

just as we found above. The binomial theorem may be expressed as

$$(a+h)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}h^k = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}h + \binom{n}{2}a^{n-2}h^2 + \cdots + \binom{n}{n-1}ah^{n-1} + \binom{n}{n}h^n.$$