College of the Holy Cross
Math 135 (Calculus I)
Supplement 1: Power Functions (September 7)

A *power function* has the form $f(x) = cx^r$, with c and r numbers, and (usually) $x > 0$. The constant c is the *coefficient*, and r is the *exponent*.

Example 1. The circumference of a circle is $f(x) = 2\pi x$. Here $c = 2\pi$ and $r = 1$. Because $r = 1$, the circumference of a circle is *proportional to* the radius.

Question: Suppose a long string were tied snug around the earth at the equator. If we make the string 30 feet longer and have it run at a fixed height above the ground, about how high off the ground is the string: Enough room for an atom? For a bacterium? For a mouse? For a person? (Answer below.)

Example 2. The volume of a solid ball of radius x is $V(x) = \frac{4}{3}\pi x^3$, while the surface area is $A(x) = 4\pi x^2$.

A spherical biological cell of radius x contains a metabolizing volume proportional to x^3, but a membrane whose area is proportional to x^2. Nutrients can enter a cell, and waste products can leave the cell, only through the membrane.

Doubling the radius multiplies the total metabolism by a factor of 8, but multiplies the membrane's area only by a factor of 4, and therefore doubles the cell's nutrient and waste-disposal requirements per unit capacity. A large cell cannot feed itself or dispose of waste products quickly enough to survive.

(Contrast this with the relative scale-invariance of many inorganic phenomena, such as mudslides after the recent earthquake in Hokkaido and rivulets left by the curb after a rain storm, or the clouds of dust in front of the Milky Way versus clouds you see in the sky.)

Example 3. To a good approximation, the apparent size (on your retina or in a photograph) of a distant object is *inversely proportional* to the distance x. For instance, the apparent height of a particular person at distance x is about $h(x) = c/x$. Thus, an object twice as far from your eye looks about half as large.

Question: The full moon is about the same apparent size as the following held at arm's length: a pea, a penny, a peach. Try to answer this by "first impression", then make an educated guess. (Answer below.)

Example 4. Any two points in the first quadrant (not on a vertical line) lie on a unique power graph. To find the power graph $y = cx^r$ passing through the points $(2, 5)$ and $(8, 40)$, we first use each point (x, y) to get an equation involving c and r:

$$5 = c \cdot 2^r, \qquad 40 = c \cdot 8^r.$$

To solve, it's usually easiest to divide one by the other to eliminate c. Here, dividing the second by the first gives $(40/5) = (8/2)^r$, or $8 = 4^r$, so $r = 3/2$. Plugging this into the first equation gives $c = 5/2^r = 5/2^{3/2}$. Our power law is therefore $y = (5/2^{3/2})x^{3/2} = 5(x/2)^{3/2}$.

Answers. If r_0 is the earth's radius in feet, the circumference, i.e., the length of the string, is $2\pi r_0$. Lengthening the string increases this to $2\pi r_0 + 30 = 2\pi(r_0 + \Delta r)$, where Δr is the height of the string from the ground. Solving for Δr gives $\Delta r = 30/(2\pi) \approx 5$: The longer string is nearly 5 feet off the ground, high enough for a person to stoop and walk beneath.

The moon is about 2200 miles in diameter, and lies about $238,000$ miles (about 100 diameters) from the earth. Your arm is about 30 inches long, so an object 1/100 that diameter, 1/3 of an inch, looks the same size at arm's length. That's "a pea".