College of the Holy Cross, Fall Semester, 2018
Math 135-06, Midterm 2 Review Sheet Selected Answers (Professor Hwang)

1. $\displaystyle\lim_{x\to 4}\frac{x^4-256}{x-4}=256$; $\displaystyle\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\frac{1}{4}$; $\displaystyle\lim_{x\to 4}\frac{\frac{1}{x}-\frac{1}{4}}{x-4}=-\frac{1}{16}$; $\displaystyle\lim_{x\to 4}\frac{\frac{1}{\sqrt{x}}-\frac{1}{2}}{x-4}=-\frac{1}{16}$.

2. (b) $\displaystyle\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}=\frac{4}{3}$.

3. $\displaystyle\lim_{h\to 0}\frac{\sin(\pi h)}{h}=\pi$; $\displaystyle\lim_{h\to 0}\frac{\sin(\pi h)}{\sin h}=\pi$; $\displaystyle\lim_{h\to 0}\frac{1-\cos h}{h^2}=\frac{1}{2}$; $\displaystyle\lim_{h\to 0}\frac{1-\cos^2 h}{h^2}=1$.

4. $\displaystyle\lim_{h\to 0}\frac{e^{3h}-1}{h}=3$; $\displaystyle\lim_{h\to 0}\frac{e^{-h}-1}{h}=-1$; $\displaystyle\lim_{h\to 0}\frac{e^{h^2}-1}{h^2}=1$; $\displaystyle\lim_{h\to 0}\frac{2^h-1}{h}=\ln 2$.

5. Give an example of:

   (a) A function f so that $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{x\to 0}|f(x)|=1$.

   (b) Functions f, g so that $\lim_{x\to 0} f(x)$, $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0} f(x)g(x)$ exists.

   Answer In each part, we may take $f(x)=g(x)=x/|x|$ for $x\neq 0$.

   (c) Functions f, g so that $\lim_{x\to 0} f(x)=\infty$, $\lim_{x\to 0} g(x)=0$, and:
   (i) $\lim_{x\to 0} f(x)g(x)=\infty$; (ii) $\lim_{x\to 0} f(x)g(x)=0$; (iii) $\lim_{x\to 0} f(x)g(x)=\pi$.

   Answer In each part, we may take $f(x)=\frac{1}{x}$. Then, for instance, (i) $g(x)=\sqrt{x}$; (ii) $g(x)=x^2$; (iii) $g(x)=\pi x$.

6. Suppose $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

   (a) Find the domain of f, and the limits of $f(x)$ as $x\to\infty$ and as $x\to-\infty$.

   Answer The domain is $(-\infty,\infty)$. The respective limits as $x\to\pm\infty$ are ± 1.

   (b) Using results we have covered, show f is increasing: If $u<v$, then $f(u)<f(v)$.

   (c) Sketch the graph of f.

   Solution We have

$$f(x)=\frac{e^{2x}-1}{e^{2x}+1}=\frac{e^{2x}+1-2}{e^{2x}+1}=1-\frac{2}{e^{2x}+1}.$$

   If $u<v$, then $e^{2u}<e^{2v}$, so

$$\frac{1}{e^{2u}+1}>\frac{1}{e^{2v}+1},\quad\text{so}\quad -\frac{2}{e^{2u}+1}<-\frac{2}{e^{2v}+1}.$$

   (In words, the denominator increases as x increases. This makes the fraction smaller in absolute value, and since the fraction is negative, "smaller absolute value" means "closer to 0", which means "larger".)

   The graph has the same general shape as $y=\dfrac{e^x}{e^x+1}$, which we examined in class.