College of the Holy Cross, Fall Semester, 2018
Math 135-06, Midterm 2 Review Sheet (Professor Hwang)

The second midterm in Math 135-06 will emphasize the material from Chapter 2 and the first two sections of Chapter 3 that we've covered in class and on supplementary notes, on the problem sets and practice questions, and on the worksheets and group work.

- Rates of change, limits of functions given by closed formulas, one-sided limits, infinite limits, limits at infinity, the squeeze theorem.
- The limits $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$, $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0$, $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$, and consequences.
- Continuous functions, removable and jump discontinuities, infinite discontinuities. The intermediate value theorem and applications.
- Difference quotients, computing derivatives as limits of difference quotients, the derivative as a function.

The questions below range from routine to challenging. They're generally harder than test questions, but use nothing beyond techniques we've covered.

1. Evaluate the limits: $\lim\limits_{x\to 4} \dfrac{x^4 - 256}{x-4}$; $\lim\limits_{x\to 4} \dfrac{\sqrt{x} - 2}{x-4}$; $\lim\limits_{x\to 4} \dfrac{\frac{1}{x} - \frac{1}{4}}{x-4}$; $\lim\limits_{x\to 4} \dfrac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4}$.

2. (a) Verify that $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$.

   (b) Use part (a) to evaluate $\lim\limits_{x\to 8} \dfrac{\sqrt[3]{x} - 2}{x-8}$.

3. Evaluate the limits: $\lim\limits_{h\to 0} \dfrac{\sin(\pi h)}{h}$; $\lim\limits_{h\to 0} \dfrac{\sin(\pi h)}{\sin h}$; $\lim\limits_{h\to 0} \dfrac{1-\cos h}{h^2}$; $\lim\limits_{h\to 0} \dfrac{1-\cos^2 h}{h^2}$.

4. Evaluate the limits: $\lim\limits_{h\to 0} \dfrac{e^{3h} - 1}{h}$; $\lim\limits_{h\to 0} \dfrac{e^{-h} - 1}{h}$; $\lim\limits_{h\to 0} \dfrac{e^{h^2} - 1}{h^2}$; $\lim\limits_{h\to 0} \dfrac{2^h - 1}{h}$.

5. Give an example of:

   (a) A function f so that $\lim\limits_{x\to 0} f(x)$ does not exist, but $\lim\limits_{x\to 0} |f(x)| = 1$.

   (b) Functions f, g so that $\lim\limits_{x\to 0} f(x)$, $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0} f(x)g(x)$ exists.

   (c) Functions f, g so that $\lim_{x\to 0} f(x) = \infty$, $\lim\limits_{x\to 0} g(x) = 0$, and:
   (i) $\lim\limits_{x\to 0} f(x)g(x) = \infty$; (ii) $\lim\limits_{x\to 0} f(x)g(x) = 0$; (iii) $\lim\limits_{x\to 0} f(x)g(x) = \pi$.

6. Suppose $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

   (a) Find the domain of f, and the limits of $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

   (b) Using results we have covered, show f is increasing: If $u < v$, then $f(u) < f(v)$.

   (c) Sketch the graph of f.

7. In each part, $f(x) = \lfloor x^2 \rfloor$ and $g(x) = \lceil x^2 \rceil$.

   (a) Sketch the graphs $y = f(x)$ and $y = g(x)$. Hint: First sketch $y = x^2$.

   (b) Find the one-sided limits of f and of g at $c = 2$; at $c = -\sqrt{2}$.

8. Let $f(x) = \frac{1}{2} \lfloor 2x \rfloor$. Sketch the graph $y = f(x)$, determine where f is discontinuous, and find the one-sided limits at the discontinuities.

9. Granted that $\ln x \to \infty$ as $x \to \infty$, what is the limit as $x \to \infty$ of $f(x) = \ln(\ln x)$, and what is the domain of f? If $f(c) = 10$, what is c? How many digits does c have? Hint: A positive real number x (not a power of 10) has $\lceil \log x \rceil$ digits.

10. (a) Find the equation of the tangent line to the parabola $y = x^2$ at (a, a^2).

    (b) If (x_0, y_0) does not lie on the parabola $y = x^2$, how many tangent lines to the parabola pass through (x_0, y_0)? (The answer depends on the point, and can be guessed easily from a sketch; give an algebraic justification using part (a).)

11. Show there are at least two positive c satisfying $c^{1000} = e^c$.

12. (a) Calculate the derivative $\cos' a = \lim\limits_{h \to 0} \dfrac{\cos(a + h) - \cos a}{h}$. Hint: Look up the formula for $\cos(x + y)$.

    (b) Find the slope-intercept equation of the line tangent to the graph $y = \cos x$ at the point $(a, \cos a)$.

    (c) Show that there is a point c between 0 and π so that the tangent line to $y = \cos x$ at $(c, \cos c)$ passes through the origin, and illustrate with a sketch.

13. Show the graph $y = e^x$ has precisely one tangent line passing through the origin.

14. Use the definition to calculate the derivative at x of $f(x) = e^x$ and $g(x) = 2^x$.

15. (a) Use the definition to calculate the derivative of $f(x) = \dfrac{\sin x}{x}$.

    (b) Show there exists a c between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $f'(c) = 0$.

16. If $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, show $f'(0)$ exists. Hint: You must use the definition (and the squeeze theorem) even if you know calculus. Why?

17. (a) If $y = f(x)$ is an invertible function, $x = g(y)$ is the inverse, and $y+k = f(x+h)$, what is the exact relationship between the difference quotients

    $$\frac{f(x + h) - f(x)}{h}, \qquad \frac{g(y + k) - g(y)}{k}?$$

    (b) Use (a) and the fact that $f(x) = e^x$ is its own derivative to find the derivative of ln at a positive number y. Hint: $y + k = e^{x+h}$ means $x + h = \ln(y + k)$.

    (c) Evaluate $\lim\limits_{h \to 0}(1 + h)^{1/h} = \lim\limits_{h \to 0} e^{\frac{1}{h}\ln(1+h)}$ and $\lim\limits_{n \to \infty}(1 + \frac{1}{n})^n = \lim\limits_{n \to \infty} e^{n \ln(1+\frac{1}{n})}$.

    (c) If $r > 0$, evaluate the continuously-compounded interest multiplier $\lim\limits_{n \to \infty}(1+\frac{r}{n})^{nt}$.