College of the Holy Cross, Fall Semester, 2018
Math 135-06, Midterm 1 Review Sheet (Professor Hwang)

The first midterm in Math 135-06 will include the material from Chapter 1 we've covered in class and on supplementary notes, on the problem sets and practice questions, and on the worksheets and group work.

- The number line; absolute value, distance; interpolating an interval: $x = a+(b-a)t$ for $0 \leq t \leq 1$.
- Functions, domains; even and odd functions; increasing and decreasing functions; graphing the sum of two known functions, or a constant multiple of a known function.
- Lines, slope; slope-intercept form, point-slope form, finding the line through two points; parallel and perpendicular lines.
- Quadratics, the quadratic formula, factoring, completing the square.
- Power functions, finding the power law between two points.
- Radians and degrees; the circular functions, especially cos, sin, tan, and sec; values of these functions on the unit circle; polar coordinates, polar graphs; inverse trig functions.
- Exponential functions with base 2, e, and 10, laws of exponents; common (base 10) and natural (base e) logarithms, properties of logs, log tables.

The questions below range from routine to challenging. They're generally harder than test questions, but use nothing beyond techniques we've covered.

1. In the interval $[a, b] = [1, \sqrt{2}]$, find:
   (a) The midpoint; (b) The point 1/3 of the way from the left; (c) The point 5/7 of the way from the left.

2. Suppose f is a function such that $f(1.02) = 5.3$ and $f(1.03) = 5.7$.

   (a) What is your best guess for $f(1.025)$? For $f(1.028)$? What assumption are you making in your guess?

   (b) Assume the graph of f is a line for $1.02 \leq x \leq 1.03$. Find a formula for $f(x)$ in this interval.

3. Let $f(x) = \frac{1}{2}\big(|x+2| + |x-2|\big)$ and $g(x) = \frac{1}{2}\big(|x+2| - |x-2|\big)$.

   (a) Sketch the graphs $y = f(x)$ and $y = g(x)$ for $-3 \leq x \leq 3$ by sketching each summand separately, then adding.

   (b) Show algebraically that f is even and g is odd; be sure this is confirmed by your graphs in (a). Hint: $|-a| = |a|$ for all real a.

   (c) Give piecewise formulas for $\frac{1}{2}|x+2|$ and $\frac{1}{2}|x-2|$.

   (d) Use part (c) to find piecewise formulas for $f(x)$ and $g(x)$; be sure your answers agree with your graphs. Hint: You'll need three separate cases for each.

4. (a) A rectangular sheet of paper (taller than its width) is cut in half by a horizontal line. If the two halves have the same aspect ratio (long-to-short edge length) as the original sheet,[1] find the proportions of the paper. Hint: Make a sketch.

   (b) A square is removed from a rectangular sheet of paper (taller than its width) by a horizontal line. If the remaining rectangle has the same aspect ratio as the original sheet, find the proportions of the paper.

5. The sum of two positive real numbers is $\sqrt{12}$. How large could their product be?

6. Find c and r if the power law $y = cx^r$ passes through:
   (a) $(3,4)$ and $(6,16)$; (b) $(3,4)$ and $(6,1)$; (c) $(3,4)$ and $(6,12)$.

7. Recall that $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$ and $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$. Calculate the exact values $\sin\frac{\pi}{8}$, $\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$. (Your answers will contain square roots of square roots.)

8. Show that: (a) $1+\tan^2\theta = \sec^2\theta$; (b) $\tan(2\theta) = 2\tan\theta/(1-\tan^2\theta)$.

9. On a single polar graph, sketch: $r = 3+\cos(3\theta)$; $r = 2+2\cos(3\theta)$; $r = 1+3\cos(3\theta)$.

10. Find a Cartesian equation for each polar graph:
    (a) $r = 2a\cos\theta$; (b) $r = a\sec\theta$; (c) $r^2 = \sec(2\theta)$.

11. If $\log x = 0.14159$ and $\log y = 0.642$, find:
    (a) $\log(1000x)$; (b) $\log(xy)$; (c) $\log(x^2/y)$; (d) $\log(x^2/y^3)$.

    (e) Use the four-place log table to estimate x; check your answer with a calculator.

12. Explain why tables of logarithms (common or natural) only cover the domain $1 \le x \le 10$.

13. Recall that and $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

    (a) Calculate $\cosh x + \sinh x$, $\cosh x - \sinh x$, and $\cosh^2 x - \sinh^2 x$.

    (b) Show that $\cosh(2x) = \cosh^2 x + \sinh^2 x$ and $\sinh(2x) = 2\sinh x\cosh x$.

    (c) Suppose $y = \sinh x$. Solve for y. Hint: Move everything to one side, and multiply by $2e^x$ to obtain a quadratic in e^x. Now use the quadratic formula.

14. Write each of the following in the form $A \times 10^n$, with $1 \le A < 10$ and n an integer: (a) $x = e^{20}$; (b) $y = e^{e^{20}}$. Hints: $\log x = 20\log e$, and similarly for y. If, for example, $\log u = 3.14159$, then $u = 10^{0.14159} \times 10^3 \approx 1.385 \times 10^3$.

Selected Answers 1. (a) $\frac{1}{2}(1+\sqrt{2})$; (b) $\frac{1}{3}(2+\sqrt{2})$; (c) $\frac{1}{7}(2+5\sqrt{2})$. 2. (b) $f(x) = 5.3 + \frac{5.7-5.3}{0.01}(x-1.02)$. (For (a), plug x into this formula.) 3. (c) $\frac{1}{2}|x+2| = \begin{cases} -\frac{1}{2}(x+2) & x < -2, \\ \frac{1}{2}(x+2) & -2 \le x; \end{cases}$ $\frac{1}{2}|x-2| = \begin{cases} \frac{1}{2}(2-x) & x < 2, \\ \frac{1}{2}(x-2) & 2 \le x. \end{cases}$ (d) $f(x) = \begin{cases} -x & x < -2, \\ 2 & -2 \le x \le 2, \\ x & 2 < x; \end{cases}$ $g(x) = \begin{cases} -2 & x < -2, \\ x & -2 \le x \le 2, \\ 2 & 2 < x. \end{cases}$
4. (a) $\sqrt{2} : 1$, (b) $\frac{1}{2}(1+\sqrt{5}) : 1$. 5. Max product = 3. (6) (a) $c = \frac{4}{9}$, $r = 2$; (b) $c = 36$, $r = -2$; (c) $c = \frac{4}{3^r}$, $r = \frac{\ln 3}{\ln 2}$. 7. (a) $\frac{1}{2}\sqrt{2-\sqrt{2}}$; (b) $\frac{1}{2}\sqrt{2+\sqrt{3}}$; (c) $\frac{1}{2}\sqrt{2-\sqrt{3}}$. 10. (a) $(x-a)^2 + y^2 = a^2$; (b) $x = a$; (c) $x^2 - y^2 = 1$. 11. (a) 3.14159; (b) 0.78359; (c) -0.35882; (d) -1.64282; (e) Linear interpolation gives $x \approx 1.385451613$. A calculator gives $x \approx 1.385447266$. 13. (a) e^x, e^{-x}, 1. (b) $x = \frac{1}{2}\ln(y + \sqrt{y^2+1})$. 14. (a) $x \approx 4.85165 \times 10^8$; (b) $y \approx 1.2795 \times 10^{2\times 10^8}$.

[1]Factlet: This is the ratio of A4 paper, which is used almost everywhere outside the United States.