

Math 135, Group Work #3 Solutions

1. In each part, $f(x) = x^2 e^{-x}$.

- (a) Calculate $f'(x)$ and find the interval(s) where f is increasing or decreasing. Explain why f has a maximum value for $x \geq 0$.
- (b) Calculate $f''(x)$ and find the interval(s) where f is convex or concave.
- (c) Use the information found to sketch the graph of f for $x \geq 0$. Be sure to include the coordinates of any interesting points.

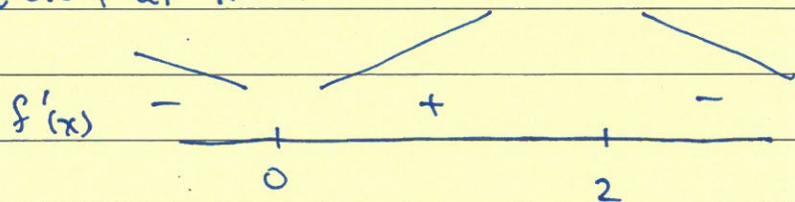
Solution

(a) By the product rule, $f'(x) = 2x e^{-x} + x^2 e^{-x}(-1) = (2x - x^2)e^{-x} = x(2-x)e^{-x}$.

Thus $f'(x) = 0$ where $x = 0$ or $x = 2$. Since $f'(-1) = -3e < 0$, f is decreasing on $(-\infty, 0]$. Since $f'(1) = e^{-1} > 0$, f is increasing on $[0, 2]$.

Finally, $f'(3) = -3e^{-3} < 0$, so f is decreasing on $[2, \infty)$.

Since f is increasing on $[0, 2]$ and decreasing on $[2, \infty)$, $f(x)$ takes a maximum value of $f(2) = 4e^{-2} \approx 0.54$ at $x = 2$.



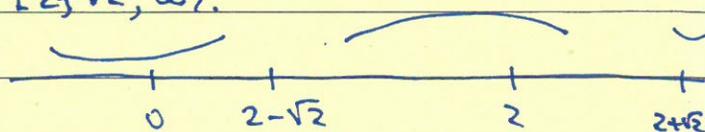
(b) Again by the product rule, $f''(x) = (2-2x)e^{-x} - (2x-x^2)e^{-x} = (2-4x+x^2)e^{-x}$.

By the quadratic formula, $f''(x) = 0$ where $x = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 2}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$.

Since $f''(0) = 2 > 0$, f is convex (concave up) on $(-\infty, 2 - \sqrt{2}]$.

Since $f''(2) = -2e^{-2} < 0$, f is concave (down) on $[2 - \sqrt{2}, 2 + \sqrt{2}]$.

Since $f''(4) = 2e^{-4} > 0$, f is convex on $[2 + \sqrt{2}, \infty)$.



(c) $f(0) = 0$ is a minimum;

$f(2) = 4e^{-2} \approx 0.54$ is a maximum.

f has inflection points at $(2 - \sqrt{2}, f(2 - \sqrt{2})) \approx (0.586, 0.191)$

and $(2 + \sqrt{2}, f(2 + \sqrt{2})) \approx (3.414, 0.384)$.

See graph paper for graph.

2. Show that the function $f(x) = \begin{cases} -1 & x < -1 \\ \frac{3}{2}x - \frac{1}{2}x^3 & -1 \leq x \leq 1 \\ 1 & 1 < x \end{cases}$

is differentiable and non-decreasing on $(-\infty, \infty)$.

Solution:

We first show f is continuous. The only points needing attention are $x=1$ and $x=-1$. We have $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \left(\frac{3}{2}x - \frac{1}{2}x^3 \right) = \frac{3}{2} - \frac{1}{2} = 1$,
 $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 1 = 1 = \lim_{x \rightarrow 1^-} f(x)$.

Since the one-sided limits of f exist at $x=1$, and are equal to $f(1)=1$, f is continuous at $x=1$. Similarly, we have

$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} -1 = -1 = f(-1) = \lim_{x \rightarrow -1^+} \left(\frac{3}{2}x - \frac{1}{2}x^3 \right) = \lim_{x \rightarrow -1^+} f(x)$, so f is

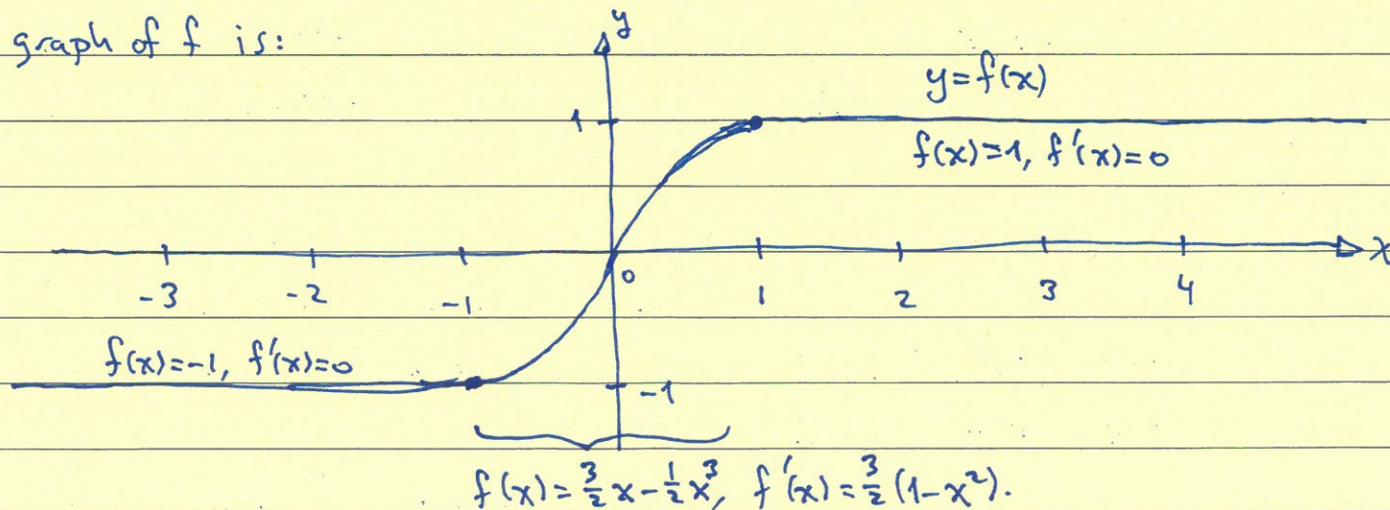
continuous at $x=-1$.

Differentiating, we have $f'(x) = \begin{cases} 0 & x < -1 \text{ or } 1 < x \\ \frac{3}{2} - \frac{3}{2}x^2 & \text{if } -1 < x < 1. \end{cases}$

Since $\lim_{x \rightarrow \pm 1} \left(\frac{3}{2} - \frac{3}{2}x^2 \right) = 0$, the one-sided limits of f' at $x=1$ exist and are equal (so f is differentiable at $x=1$), and the one-sided limits of f' at $x=-1$ exist and are equal (so f is differentiable at $x=-1$).

In summary, $f'(x) = 0$ if $x \leq -1$ or $1 \leq x$, and $f'(x) = \frac{3}{2} - \frac{3}{2}x^2 = \frac{3}{2}(1-x^2)$ if $-1 < x < 1$. Since f is differentiable on $(-\infty, \infty)$ and $f'(x) \geq 0$, f is non-decreasing.

The graph of f is:



3. Suppose a and b are constants, and that $f(x) = \begin{cases} ax+b & x < 1 \\ x+x^2 & 1 \leq x \end{cases}$.

- (a) Find all values of a and b so that f is continuous on $(-\infty, \infty)$.
- (b) Find all values of a and b so that f is differentiable on $(-\infty, \infty)$.
- (c) On a single graph, sketch $y=f(x)$ if f is continuous at $a=-1, 0, \text{ or } 1$.

Solution

- (a) f is clearly continuous except possibly at $x=1$. For continuity at 1, we require $a+b = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 2$, or $a+b=2$.
- (b) f is clearly differentiable except possibly at $x=1$. In addition to continuity, we require $a = \lim_{x \rightarrow 1^-} a = \lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^+} 1+2x = 3$. That is f is differentiable if, and only if, $a=3$ and $b=-1$.
- (c) See graph paper.

4. Suppose a and b are constants, and that $f(x) = \begin{cases} 2x-x^2 & x < 2 \\ a+\frac{b}{x} & 2 \leq x \end{cases}$.

- (a) Find all values of a and b so that f is continuous on $(-\infty, \infty)$.
- (b) Find all values of a and b so that f is differentiable on $(-\infty, \infty)$.

Solution...

- (a) f is clearly continuous except possibly at $x=2$. For continuity at $x=2$, we require $-2 = \lim_{x \rightarrow 2^-} 2x-x^2 = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = a + \frac{b}{2}$. Thus f is continuous at $x=2$, and therefore on $(-\infty, \infty)$ if and only if $a + \frac{b}{2} = -2$.
- (b) f is clearly differentiable except possibly at $x=2$. In addition to continuity, we require $-2 = \lim_{x \rightarrow 2^-} 2-2x = \lim_{x \rightarrow 2^-} f'(x) = \lim_{x \rightarrow 2^+} f'(x) = \lim_{x \rightarrow 2^+} -\frac{b}{x^2} = -\frac{b}{4}$, or $b=8$, and thus $a=-4$.