College of the Holy Cross
Math 135 (Calculus I)
Group Work 2 Solutions: Tangent Lines and the IVT

1. Let a stand for an arbitrary real number.

(a) Find the slope-intercept equation of the line tangent to the parabola $y = x^2$ at the point where $x = a$.
Solution The point of tangency is (a, a^2). At this point, the slope is

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0}(2a + h) = 2a.$$

The point-slope equation of the tangent line is $y - y_0 = m(x - x_0)$, or

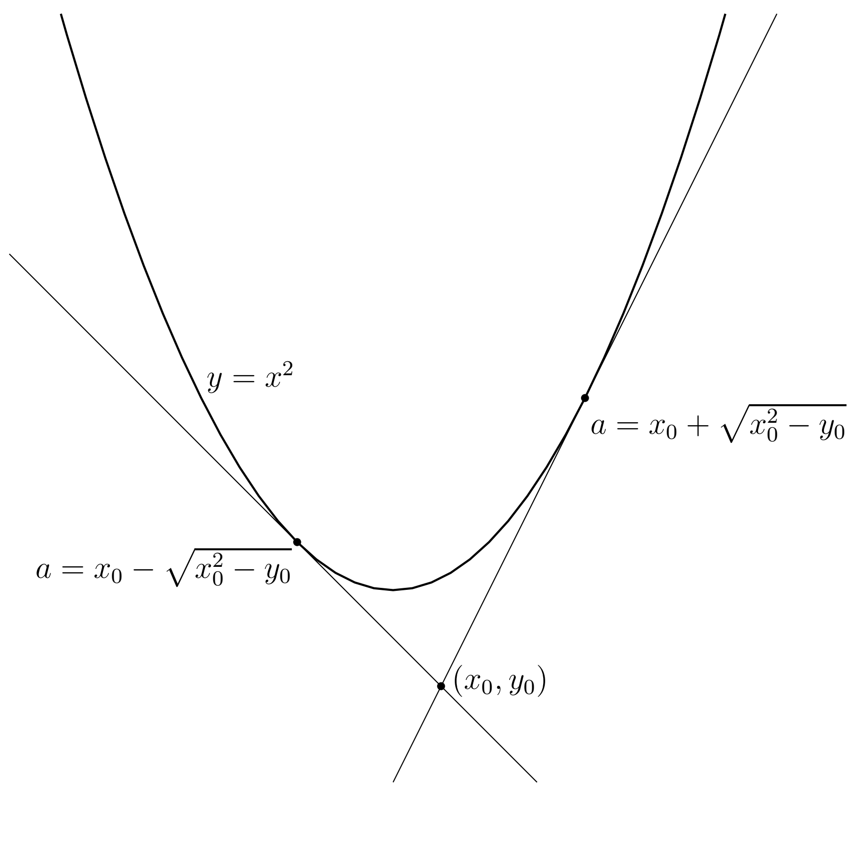
$$y - a^2 = 2a(x - a) = 2ax - 2a^2.$$

The slope-intercept form is therefore $y = 2ax - a^2$.

(b) Which points (x_0, y_0) in the plane lie on some tangent line to the parabola $y = x^2$? For these points, how many distinct tangent lines contain the point?
Solution The point (x_0, y_0) lies on a tangent line to the parabola precisely when there is a real a satisfying $y_0 = 2ax_0 - a^2$, or $a^2 - (2x_0)a + y_0 = 0$. The quadratic formula gives

$$a = \frac{2x_0 \pm \sqrt{(2x_0)^2 - 4y_0}}{2} = x_0 \pm \sqrt{x_0^2 - y_0}.$$

There are two real solutions if $0 < x_0^2 - y_0$, i.e., if $y_0 < x_0^2$, so (x_0, y_0) lies below the parabola.
There is one real solution if $x_0^2 - y_0 = 0$, i.e., (x_0, y_0) lies on the parabola.
There are no real solutions if $x_0^2 - y_0 < 0$, i.e., (x_0, y_0) lies above the parabola.



2. Each part refers to the graph $y = \cos x$, and a stands for an arbitrary real number.

(a) Calculate the derivative $\cos' a = \lim_{h\to 0} \dfrac{\cos(a+h) - \cos a}{h}$ from the definition.

Hint: For all real a and b, we have $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

Solution Since $\lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$, we have

$$\begin{aligned}\cos'(a) &= \lim_{h\to 0} \frac{\cos(a+h) - \cos a}{h} = \lim_{h\to 0} \frac{\cos a \cos h - \sin a \sin h - \cos a}{h} \\ &= \cos a \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin a \lim_{h\to 0} \frac{\sin h}{h} = -\sin a.\end{aligned}$$

(b) Find the slope-intercept equation of the line tangent to the graph $y = \cos x$ at the point where $x = a$.

Solution Where $x = a$, the tangent line passes through the point $(a, \cos a)$ and has slope $m = -\sin a$. The point-slope equation of the tangent line is $y - \cos a = -\sin a(x - a) = -x \sin a + a \sin a$.

The slope-intercept form is $y = -x \sin a + a \sin a + \cos a$.

(c) Show that there is a positive real number $c < \pi$ so that the tangent line in part (b) passes through the origin. (5-point bonus if you locate c to within $\pi/12$ without a calculator.)

Solution We want to show there is a $c < \pi$ such that the y-intercept $\phi(c) = c \sin c + \cos c = 0$. We have $\phi(\frac{\pi}{2}) = \frac{\pi}{2} > 0$ and $\phi(\pi) = -1 < 0$. Since ϕ is a continuous function of a, the intermediate value theorem guarantees there is a c with $\frac{\pi}{2} < c < \pi$ and $\phi(c) = 0$.

To locate c more precisely, we'll try bisection or other subdivision. Since

$$\phi(\tfrac{3\pi}{4}) = \tfrac{3\pi}{4}(\tfrac{\sqrt{2}}{2}) - \tfrac{\sqrt{2}}{2} = (\tfrac{3\pi}{4} - 1)\tfrac{\sqrt{2}}{2} > (\tfrac{9}{4} - 1)\tfrac{\sqrt{2}}{2} > 0,$$

there is a solution between $\frac{3\pi}{4}$ and π. Bisecting again would force us to evaluate sin and cos at $\frac{7\pi}{8}$, so instead we'll try $\frac{5\pi}{6}$: We have

$$\phi(\tfrac{5\pi}{6}) = \tfrac{5\pi}{6} \cdot \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} = \tfrac{5\pi - 6\sqrt{3}}{6\cdot 2}.$$

Since $3 < \pi$ and $\sqrt{3} < 2$, we have $5\pi - 6\sqrt{3} > 5\cdot 3 - 6\cdot 2 = 15 - 12 = 3 > 0$. Thus $\phi(\frac{5\pi}{6}) > 0$, so there is a solution between $\frac{5\pi}{6}$ and π. The midpoint, $c = \frac{11\pi}{12}$, is accurate to within $\frac{\pi}{12}$.