College of the Holy Cross
Math 135 (Calculus I)
Group Work 2: Tangent Lines and the IVT
Due Monday, October 22

Write up your answers neatly, using algebraic calculations and complete sentences as appropriate, on a separate sheet of paper. **Readability will count for 10% of the score.** A sample (whose mathematical details differ, but whose ideas are similar) is included below to give an idea of the style and level of detail I'm hoping for. (Please write in your own voice; don't mimic my wording!)

1. Let a stand for an arbitrary real number.

(a) Find the slope-intercept equation of the line tangent to the parabola $y = x^2$ at the point where $x = a$.

(b) Which points (x_0, y_0) in the plane lie on some tangent line to the parabola $y = x^2$? For these points, how many distinct tangent lines contain the point?

2. Each part refers to the graph $y = \cos x$, and a stands for an arbitrary real number.

(a) Calculate the derivative $\cos' a = \lim\limits_{h \to 0} \dfrac{\cos(a + h) - \cos a}{h}$ from the definition.

Hint: For all real a and b, we have $\cos(a + b) = \cos a \cos b - \sin a \sin b$.

(b) Find the slope-intercept equation of the line tangent to the graph $y = \cos x$ at the point where $x = a$.

(c) Show that there is a positive real number $c < \pi$ so that the tangent line in part (b) passes through the origin. (5-point bonus if you locate c to within $\pi/12$ without a calculator.)

Sample Let a stand for an arbitrary real number.

(a) Find the slope-intercept equation of the line tangent to the graph $y = x^3$ at the point where $x = a$.

Solution If $f(x) = x^3$, then

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

The line tangent to $y = x^3$ at the point $(x_0, y_0) = (a, a^3)$ therefore has slope $m = f'(a) = 3a^2$. The point-slope equation $y - y_0 = m(x - x_0)$ of this tangent line is $y - a^3 = 3a^2(x - a)$. Expanding the right side and rearranging gives the slope-intercept equation

$$y = 3a^2x - 2a^3.$$

(b) Which points (x_0, y_0) in the plane lie on some tangent line to the curve $y = x^3$?

Solution The point (x_0, y_0) lies on the tangent line at (a, a^3) if and only if $y_0 = 3a^2x_0 - 2a^3$, if and only if $2a^3 - 3a^2x_0 + y_0 = 0$.

For a fixed (x_0, y_0), let $\phi(a) = 2a^3 - 3a^2x_0 + y_0 = a^3(2 - 3x_0/a + y_0/a^3)$. The expression in parentheses approaches 2 as $|a| \to \infty$. As a result, $\phi(a) \to \infty$ as $a \to \infty$ and $\phi(a) \to -\infty$ as $a \to -\infty$. Since ϕ is a polynomial function and therefore continuous, the Intermediate Value Theorem guarantees that $\phi(c) = 0$ for some real c. That is, every point (x_0, y_0) in the Cartesian plane lies on some tangent line to the graph $y = x^3$.