is denoted by $f^{(k)}(x)$. Thus

$$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + 2a_{n-2}x + a_{n-1}, \quad (6)$$
$$f''(x) = n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} + \cdots + 2a_n - 2, \quad (7)$$

etc. Hence we have

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{1\cdot 2} + f'''(x)\frac{h^3}{1\cdot 2\cdot 3} \quad (8)$$
$$+ \cdots + f^{(r)}(x)\frac{h^r}{r!} + \cdots + f^{(n)}(x)\frac{h^n}{n!},$$

where $r!$ is the symbol, read r *factorial*, for the product $1\cdot 2\cdot 3\cdots(r-1)r$. Here r is a positive integer, but we include the case $r = 0$ by the definition, $0! = 1$.

This formula (8) is known as *Taylor's theorem* for the present case of a polynomial $f(x)$ of degree n. We call $f'(x)$ the (*first*) *derivative of* $f(x)$, and $f''(x)$ the *second derivative* of $f(x)$, etc. Concerning the fact that $f''(x)$ is equal to the first derivative of $f'(x)$ and that, in general, the kth derivative $f^{(k)}(x)$ of $f(x)$ is equal to the first derivative of $f^{(k-1)}(x)$, see Exs. 6–9 of the next set.

In view of (8), the limit of (4) as h approaches zero is $f'(x)$. Hence $f'(x)$ *is the slope of the tangent to the graph of* $y = f(x)$ *at the point* (x, y).

In (5) and (6), let every a be zero except a_0. Thus the derivative of a_0x^n is na_0x^{n-1}, and hence is obtained by multiplying the given term by its exponent n and then diminishing its exponent by unity. For example, the derivative of $2x^3$ is $6x^2$.

Moreover, the derivative of $f(x)$ is equal to the sum of the derivatives of its separate terms. Thus the derivative of $x^3 + 4x^2 - 11$ is $3x^2 + 8x$, as found also in § 55.

EXERCISES

1. Show that the slope of the tangent to $y = 8x^3 - 22x^2 + 13x - 2$ at (x, y) is $24x^2 - 44x + 13$, and that the bend points are $(0.37, 0.203)$, $(1.46, -5.03)$, approximately. Draw the graph.

2. Prove that the bend points of $y = x^3 - 2x - 5$ are $(.82, -6.09)$, $(-.82, -3.91)$, approximately. Draw the graph and locate the real roots.

3. Find the bend points of $y = x^3 - 6x^2 + 8x + 8$. Locate the real roots.

4. Locate the real roots of $f(x) = x^4 + x^3 - x - 2 = 0$.

   Hints: The abscissas of the bend points are the roots of $f'(x) = 4x^3 + 3x^2 - 1 = 0$. The bend points of $y = f'(x)$ are $(0, -1)$ and $(-\frac{1}{2}, -\frac{3}{4})$, so that $f'(x) = 0$ has a single real root (it is just less than $\frac{1}{2}$). The single bend point of $y = f(x)$ is $(\frac{1}{2}, -\frac{37}{16})$, approximately.