The latter may be solved as a cubic equation for k^2. Any root $k^2 \neq 0$ gives a pair of quadratic factors of (21):

$$z^2 \pm 2kz + \tfrac{1}{2}q + 2k^2 \mp \frac{r}{4k}. \tag{23}$$

The four roots of these two quadratic functions are the four roots of (21). This method of Descartes (1596–1650) therefore succeeds unless every root of (22) is zero, whence $q = s = r = 0$, so that (12) is the trivial equation $z^4 = 0$.

For example, consider $z^4 - 3z^2 + 6z - 2 = 0$. Then (22) becomes

$$64k^6 - 3 \cdot 32k^4 + 4 \cdot 17k^2 - 36 = 0.$$

The value $k^2 = 1$ gives the factors $z^2 + 2z - 1$, $z^2 - 2z + 2$. Equating these to zero, we find the four roots $-1 \pm \sqrt{2}$, $1 \pm \sqrt{-1}$.

52. Symmetrical Form of Descartes' Solution. To obtain this symmetrical form, we use all three roots k_1^2, k_2^2, k_3^2 of (22). Then

$$k_1^2 + k_2^2 + k_3^2 = -\tfrac{1}{2}q, \qquad k_1^2 k_2^2 k_3^2 = \frac{r^2}{64}.$$

It is at our choice as to which square root of k_1^2 is denoted by $+k_1$ and which by $-k_1$, and likewise as to $\pm k_2$, $\pm k_3$. For our purposes any choice of these signs is suitable provided the choice give

$$k_1 k_2 k_3 = -\frac{r}{8}. \tag{24}$$

Let $k_1 \neq 0$. The quadratic function (23) is zero for $k = k_1$ if

$$(z \pm k_1)^2 = -\frac{q}{2} - k_1^2 \pm \frac{r}{4k_1} = k_2^2 + k_3^2 \mp \frac{8k_1k_2k_3}{4k_1} = (k_2 \mp k_3)^2.$$

Hence the four roots of the quartic equation (21) are

$$k_1 + k_2 + k_3, \qquad k_1 - k_2 - k_3, \qquad -k_1 + k_2 - k_3, \qquad -k_1 - k_2 + k_3. \tag{25}$$

EXERCISES

1. Solve Exs. 4, 5 of § 48 by the method of Descartes.

2. By writing y_1, y_2, y_3 for the roots k_1^2, k_2^2, k_3^2 of

$$64y^3 + 32qy^2 + 4(q^2 - 4s)y - r^2 = 0, \tag{26}$$

show that the four roots of (21) are the values of

$$z = \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3} \tag{27}$$