By § 8, any number has three cube roots, two of which are the products of the remaining one by the imaginary cube roots of unity:

$$\omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i, \qquad \omega^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i. \tag{8}$$

We can choose particular cube roots

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \qquad B = \sqrt[3]{-\frac{q}{2} - \sqrt{R}}, \tag{9}$$

such that $AB = -p/3$, since the product of the numbers under the cube root radicals is equal to $(-p/3)^3$. Hence the six values of z are

$$A, \quad \omega A, \quad \omega^2 A, \quad B, \quad \omega B, \quad \omega^2 B.$$

These can be paired so that the product of the two in each pair is $-p/3$.

$$AB = -\frac{p}{3}, \qquad \omega A \cdot \omega^2 B = -\frac{p}{3}, \qquad \omega^2 A \cdot \omega B = -\frac{p}{3}.$$

Hence with any root z is paired a root equal to $-p/(3z)$. By (5), the sum of the two is a value of y. Hence the three values of y are

$$y_1 = A + B, \qquad y_2 = \omega A + \omega^2 B, \qquad y_3 = \omega^2 A + \omega B. \tag{10}$$

It is easy to verify that these numbers are actually roots of (2). For example, since $\omega^3 = 1$, the cube of y_2 is

$$A^3 + B^3 + 3\omega A^2 B + 3\omega^2 A B^2 = -q - p(\omega A + \omega^2 B) = -q - py_2,$$

by (9) and $AB = -p/3$.

The numbers (10) are known as *Cardan's formulas* for the roots of a reduced cubic equation (2). The expression $A + B$ for a root was first published by Cardan in his *Ars Magna* of 1545, although he had obtained it from Tartaglia under promise of secrecy.

Example. Solve $y^3 - 15y - 126 = 0$.

Solution. The substitution (5) is here $y = z + 5/z$. We get

$$z^6 - 126z^3 + 125 = 0, \qquad z^3 = 1 \text{ or } 125.$$

The pairs of values of z whose product is 5 are 1 and 5, ω and $5\omega^2$, ω^2 and 5ω. Their sums 6, $\omega + 5\omega^2$, and $\omega^2 + 5\omega$ give the three roots.

EXERCISES

Solve the equations:

1. $y^3 - 18y + 35 = 0$.
2. $x^3 + 6x^2 + 3x + 13 = 0$.
3. $y^3 - 2y + 4 = 0$.
4. $28x^3 + 9x^2 - 1 = 0$.