Let us now transform this expression into polar coördinates by means of the substitutions

$$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$

Then

$$\begin{aligned} dx &= -\rho\sin\theta\, d\theta + \cos\theta\, d\rho, \\ dy &= \rho\cos\theta\, d\theta + \sin\theta\, d\rho, \end{aligned}$$

and we have

$$\begin{aligned} [dx^2 + dy^2]^{\frac{1}{2}} &= [(-\rho\sin\theta\, d\theta + \cos\theta\, d\rho)^2 + (\rho\cos\theta\, d\theta + \sin\theta\, d\rho)^2]^{\frac{1}{2}} \\ &= [\rho^2\, d\theta^2 + d\rho^2]^{\frac{1}{2}}. \end{aligned}$$

If the equation of the curve is

$$\rho = f(\theta),$$

then

$$d\rho = f'(\theta)\, d\theta = \frac{d\rho}{d\theta}\, d\theta.$$

Substituting this in the above differential expression, we get

$$\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta.$$

If then α and β are the limits of the independent variable θ corresponding to the limits in (A) and (B), p. 374, we get the **formula for the length of the arc**,

$$s = \int_\alpha^\beta \left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta, \tag{A}$$

where ρ and $\dfrac{d\rho}{d\theta}$ in terms of θ must be substituted from the equation of the given curve.

In case it is more convenient to use ρ as the independent variable, and the equation is in the form

$$\theta = \phi(\rho),$$

then

$$d\theta = \phi'(\rho)\, d\rho = \frac{d\theta}{d\rho}\, d\rho.$$

Substituting this in

$$[\rho^2\, d\theta^2 + d\rho^2]^{\frac{1}{2}}$$

gives

$$\left[\rho^2 \left(\frac{d\theta}{d\rho}\right)^2 + 1\right]^{\frac{1}{2}} d\rho.$$

Hence if ρ_1 and ρ_2 are the corresponding limits of the independent variable ρ, we get the **formula for the length of the arc**,

$$s = \int_{\rho_1}^{\rho_2} \left[\rho^2 \left(\frac{d\theta}{d\rho}\right)^2 + 1\right]^{\frac{1}{2}} d\rho, \tag{B}$$

where $\frac{d\theta}{d\rho}$ in terms of ρ must be substituted from the equation of the given curve.