Third Step. Applying the Fundamental Theorem,

$$\lim_{n=\infty}\sum_{i=1}^{n}\tfrac{1}{2}\rho_i^2\,\Delta\theta_i = \int_\alpha^\beta \tfrac{1}{2}\,\rho^2\,d\theta.$$

Hence the area swept over by the radius vector of the curve in moving from the position OP_1 to the position OD is given by the formula

$$\text{area} = \tfrac{1}{2}\int_\alpha^\beta \rho^2\,d\theta, \qquad \text{(A)}$$

the value of ρ in terms of θ being substituted from the equation of the curve.

Illustrative Example 1. Find the entire area of the lemniscate $\rho^2 = a^2\cos 2\theta$.

Solution. Since the figure is symmetrical with respect to both OX and OY, the whole area $= 4$ times the area of OAB.

Since $\rho = 0$ when $\theta = \dfrac{\pi}{4}$, we see that if θ varies from 0 to $\dfrac{\pi}{4}$, the radius vector OP sweeps over the area OAB. Hence, substituting in (A),

$$\begin{aligned}\text{entire area} &= 4\times\text{area } OAB = 4\cdot\frac{1}{2}\int_\alpha^\beta \rho^2\,d\theta\\ &= 2a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta = a^2;\end{aligned}$$

that is, the area of both loops equals the area of a square constructed on OA as one side.

EXAMPLES

1. Find the area swept over in one revolution by the radius vector of the spiral of Archimedes, $\rho = a\theta$, starting with $\theta = 0$. How much additional area is swept over in the second revolution? *Ans.* $\dfrac{4\pi^3a^2}{3}$; $8\pi^3a^2$.

2. Find the area of one loop of the curve $\rho = a\cos 2\theta$. *Ans.* $\dfrac{\pi a^2}{8}$.

3. Show that the entire area of the curve $\rho = a\sin 2\theta$ equals one half the area of the circumscribed circle.

4. Find the entire area of the cardioid $\rho = a(1-\cos\theta)$.
   Ans. $\dfrac{3\pi a^2}{2}$; that is, six times the area of the generating circle.

5. Find the area of the circle $\rho = a\cos\theta$. *Ans.* $\dfrac{\pi a^2}{4}$.

6. Prove that the area of the three loops of $\rho = a \sin 3\theta$ equals one fourth of the area of the circumscribed circle.

7. Prove that the area generated by the radius vector of the spiral $\rho = e^{\theta}$ equals one fourth of the area of the square described on the radius vector.

8. Find the area of that part of the parabola $\rho = a \sec^2 \dfrac{\theta}{2}$ which is intercepted between the curve and the latus rectum. *Ans.* $\dfrac{8a^2}{3}$.

9. Show that the area bounded by any two radii vectors of the hyperbolic spiral $\rho\theta = a$ is proportional to the difference between the lengths of these radii.