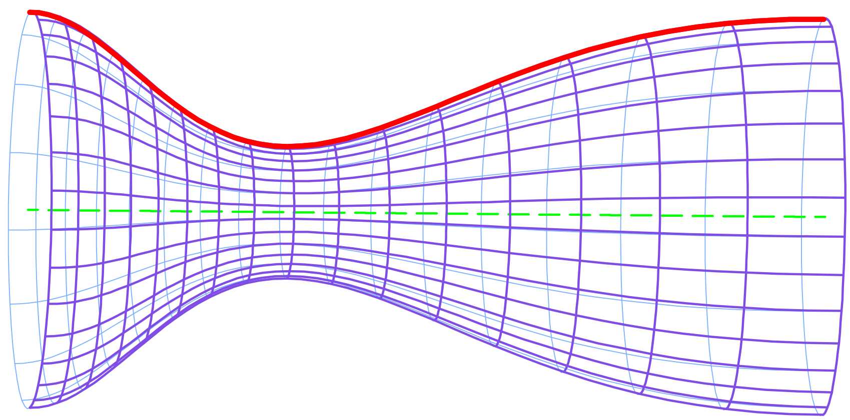
The Geometry of Surfaces of Revolution



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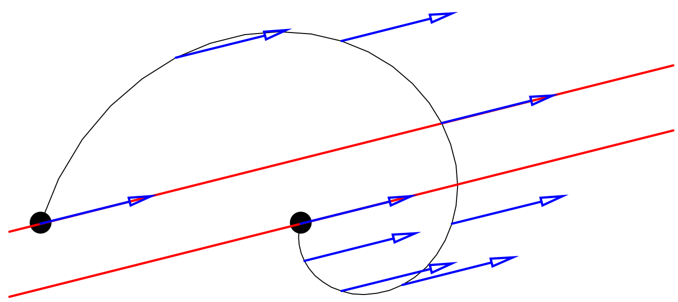
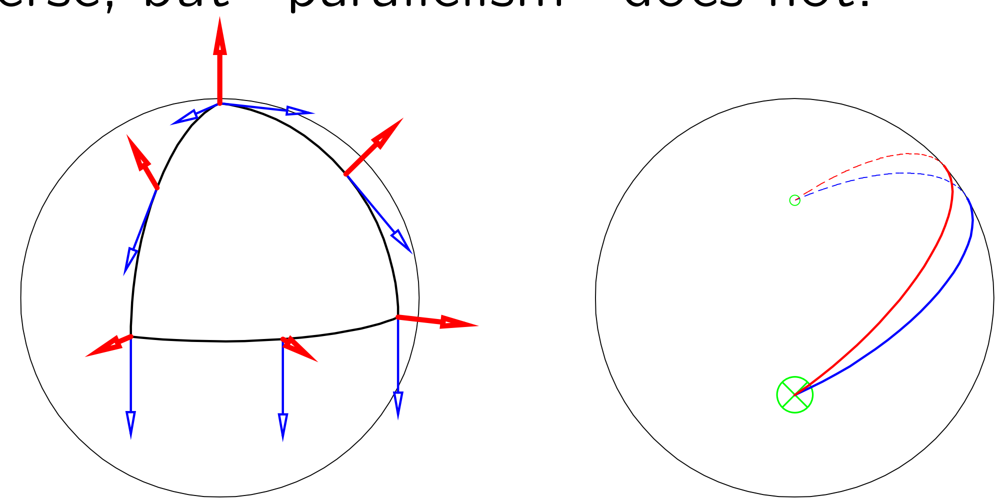
Life in a plane: Fundamental objects are points and lines, basic geometric quantities are length (or distance) and angle, $\rightsquigarrow$ area, congruence.

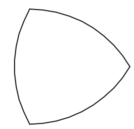
Less obvious property: Absolute parallelism.

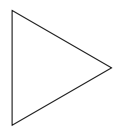


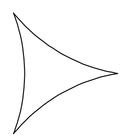
Length and angle make sense in a spherical universe, but "parallelism" does not:



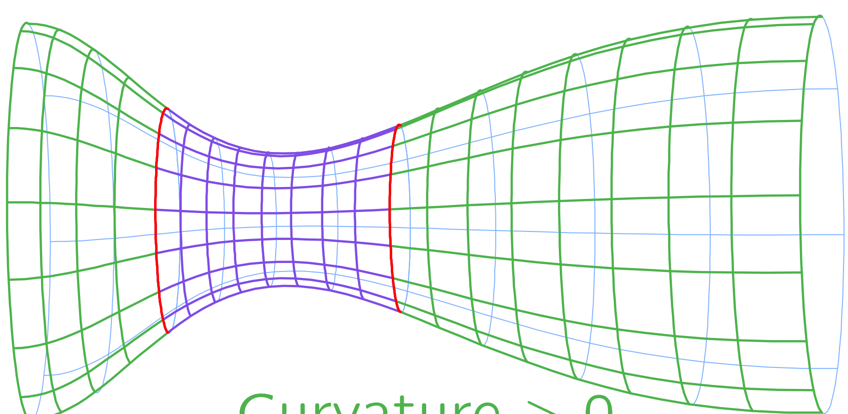
In a general surface, nearby points are joined by paths of shortest length (*geodesics*). Curvature is manifested as angular defect in small geodesic triangles (Gauss)

Excess angle $\leftrightarrow$ positive curvature 

Angle π $\leftrightarrow$ zero curvature 

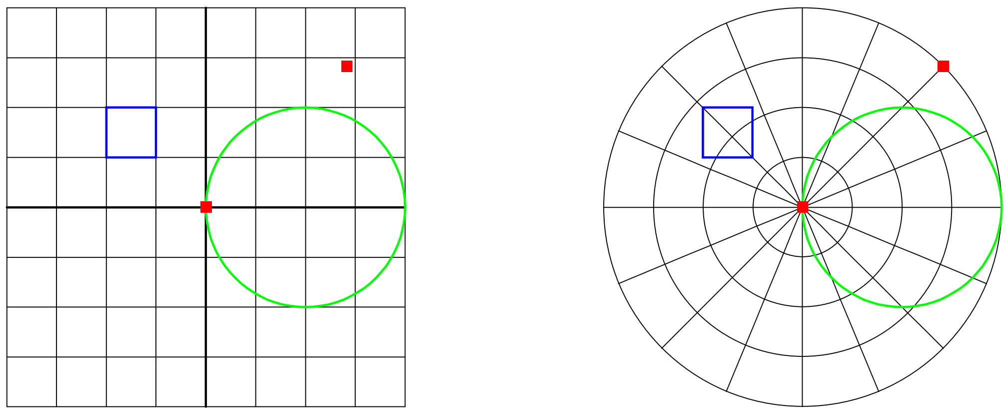
Deficient angle $\leftrightarrow$ negative curvature 

In vivo:



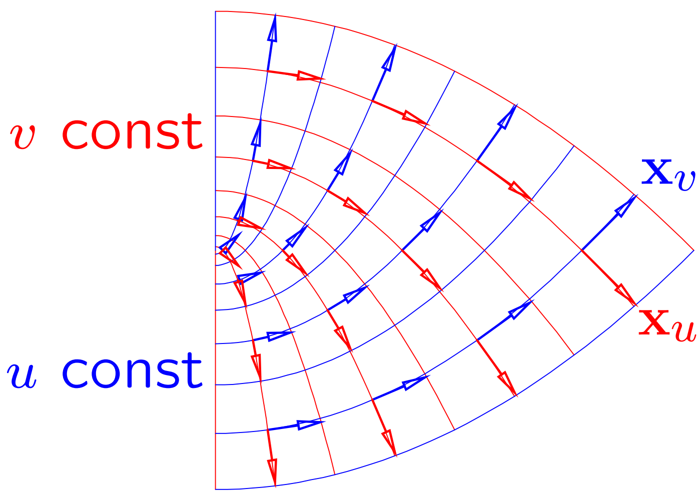
Curvature > 0
Curvature < 0

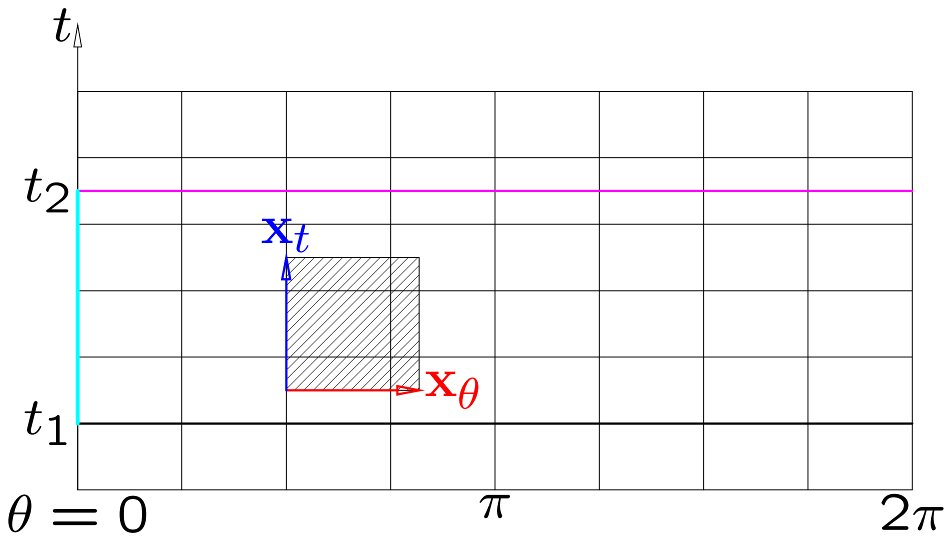
A coordinate system is a way of turning geometry (reality) into algebra (mathematics)



(u, v) coordinates on a planar region. Geometry in these coordinates is encoded in length and angle of "infinitesimal coordinate vectors"

$$\mathbf{x}_u = \frac{\partial}{\partial u}, \qquad \mathbf{x}_v = \frac{\partial}{\partial v}$$





Geometry of surface determined by the *metric*:
$E(t) = |\mathbf{x}_t|^2$, $G(t) = |\mathbf{x}_\theta|^2$
Rotational symmetry $\leftrightarrow$ E, G indep of θ

Arc length: $s = \int_{t_1}^{t_2} \sqrt{E(u)}\, du$

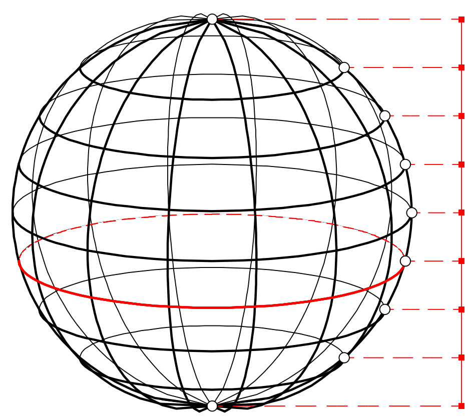
Circumference: $2\pi|\mathbf{x}_\theta| = 2\pi\sqrt{G(t)}$

Zonal area: $2\pi \int_{t_1}^{t_2} \sqrt{EG(u)}\, du$

Surface of revolution obtained by revolving a plane curve ("geometric profile") about an axis. Each point sweeps an "orbit" .

Method I: Revolve a graph, e.g. $f(t) = \sqrt{1-t^2}$.

At position t on axis, circumference is $2\pi f(t)$



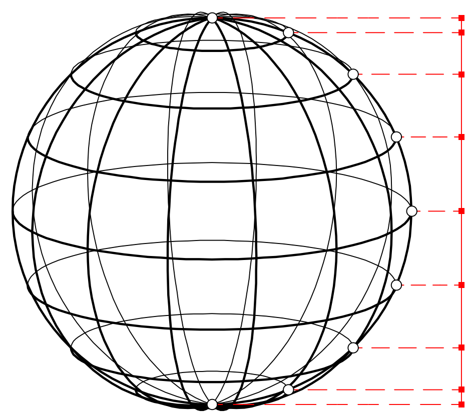
Height-angle coordinates $(-1 \le t \le 1)$

$$(x, y, z) = (t, \sqrt{1-t^2}\cos\theta, \sqrt{1-t^2}\sin\theta)$$

$E(t) = 1/\sqrt{1-t^2}$, $G(t) = \sqrt{1-t^2}$

Method II: Revolve a parametric curve, e.g. $\mathbf{c}(\phi) = (\sin\phi, \cos\phi) = \Big(t(\phi), r(\phi)\Big)$.

At parameter ϕ, position on the axis is $t(\phi)$, circumference is $2\pi r(\phi)$



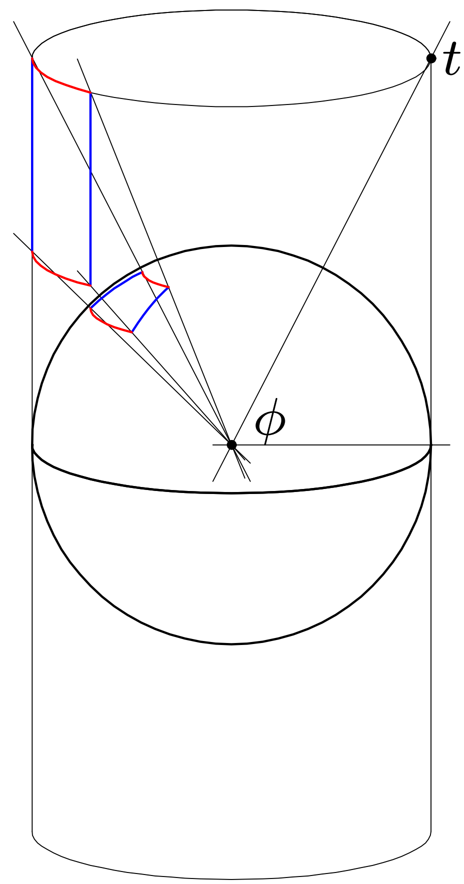
Geographic coordinates $(-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2})$

$$(x, y, z) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$$

$E(\phi) = 1$, $G(\phi) = |\cos\phi|$.

Generally, may choose the parameter ϕ to be arc length along the profile.

Mercator projection:



$$(x, y, z) = \frac{1}{\sqrt{1+t^2}}(\cos\theta, \sin\theta, t)$$

$$E(t) = \frac{1}{1+t^2} = G(t)$$

By changing the radial coordinate $t = t(r)$, we can make E and G satisfy conditions.

- Isothermal coordinates: If

$$dr = \sqrt{\frac{G(t)}{E(t)}}\, dt,$$

  then $\boxed{E(r) = G(r)}$.
  Metric angles same as coordinate angles.

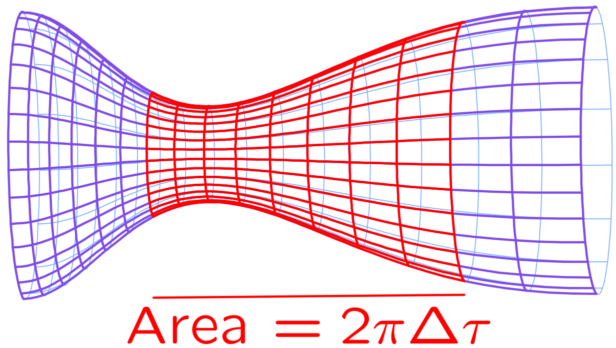
- Symplectic coordinates: If

$$dr = \sqrt{EG(t)}\, dt,$$

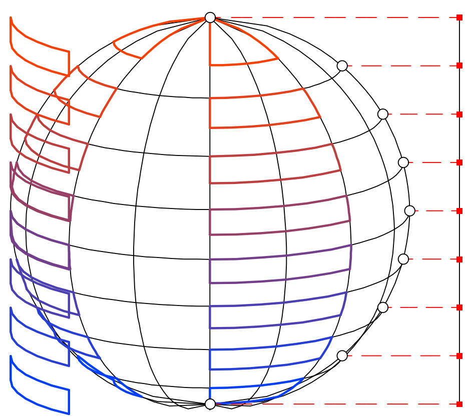
  then $\boxed{E(r) = 1/G(r)}$.
  Metric area same as coordinate area.

Mercator coordinates: isothermal
Height-angle coordinates: symplectic
Plane Cartesian coordinates: both

Method III: (Reverse description) Fix an orbit, let $2\pi\tau$ be area of zone, $2\pi\sqrt{\varphi(\tau)}$ the corresponding circumference



Remarkable example (Archimedes): Zones of equal height on unit sphere have equal area, $\varphi(\tau) = 1 - \tau^2$



$\boxed{\text{The Momentum Construction}}$

Key observation: If φ is positive near 0, there is a unique symplectic metric for which $G = \varphi$.

Define $t = \displaystyle\int_0^\tau \frac{du}{\varphi(u)}$.

Pleasant calculus exercises:

τ is a radial coordinate, $E(\tau) = 1/\varphi(\tau)$.

Area of zone $0 \le \tau$: $2\pi\tau$

Circumference of orbit: $2\pi\sqrt{\varphi(\tau)}$

Distance function: $s(t) = \displaystyle\int_0^\tau \frac{du}{\sqrt{\varphi(u)}}$

$\varphi(\tau) = 1$ ($\tau = t$, cylinder)

$\varphi(\tau) = 2\tau$ ($\tau = e^{2t}$, plane)

$$s(t) = \int_0^\tau \frac{du}{\sqrt{\varphi(u)}} = \int_0^\tau \frac{du}{\sqrt{2u}} = \sqrt{2\tau}$$

Note: $\pi s(t)^2 = 2\pi\tau$

$\varphi(\tau) = 1 - \tau^2$ ($\tau = \tanh t$, sphere)

$$s(t) = \int_0^\tau \frac{du}{\sqrt{\varphi(u)}} = \int_0^\tau \frac{du}{\sqrt{1-u^2}} = \arcsin \tau$$

Radius of corresponding orbit: $\sqrt{\varphi(\tau)} = \cos s$

How to visualize the surface defined by φ?

- Embeds iff $|\varphi'| \leq 2$

- Closes up iff $\varphi = 0$

- Cone angle is $\pi|\varphi'|$

- Finite area iff τ bounded

- Finite length iff $\displaystyle\int_0^\tau \frac{du}{\sqrt{\varphi(u)}} < \infty$

Weird examples:

$\varphi(\tau) = 2\tau + \tau^3$, $0 \leq \tau$

$\varphi(\tau) = 2\tau/(1-\tau)$, $0 \leq \tau < 1$

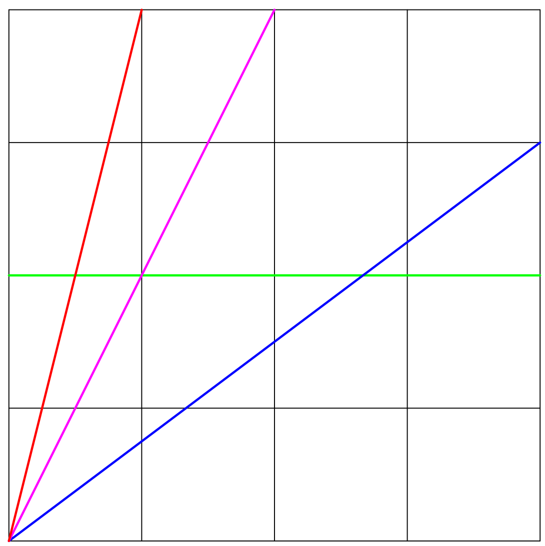
Area? Distance to edge? Length of orbits?

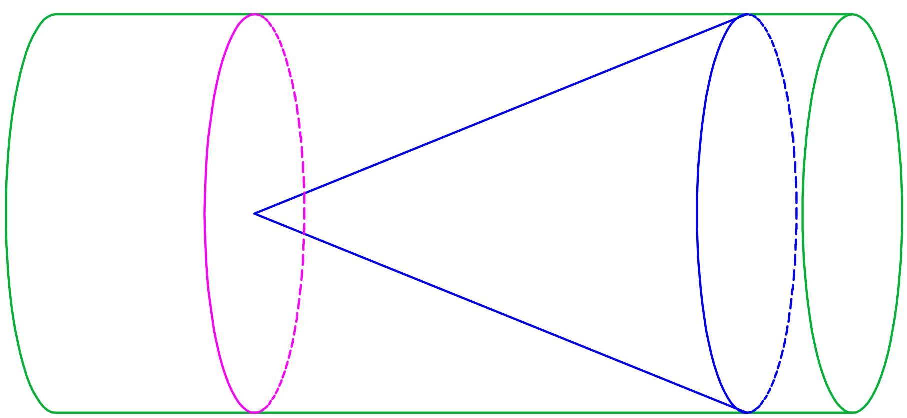
Is this description just a curiosity?

No: The (Gaussian) curvature is $\kappa = -\frac{1}{2}\varphi''(\tau)$,
Gauss-Bonnet is the FTC

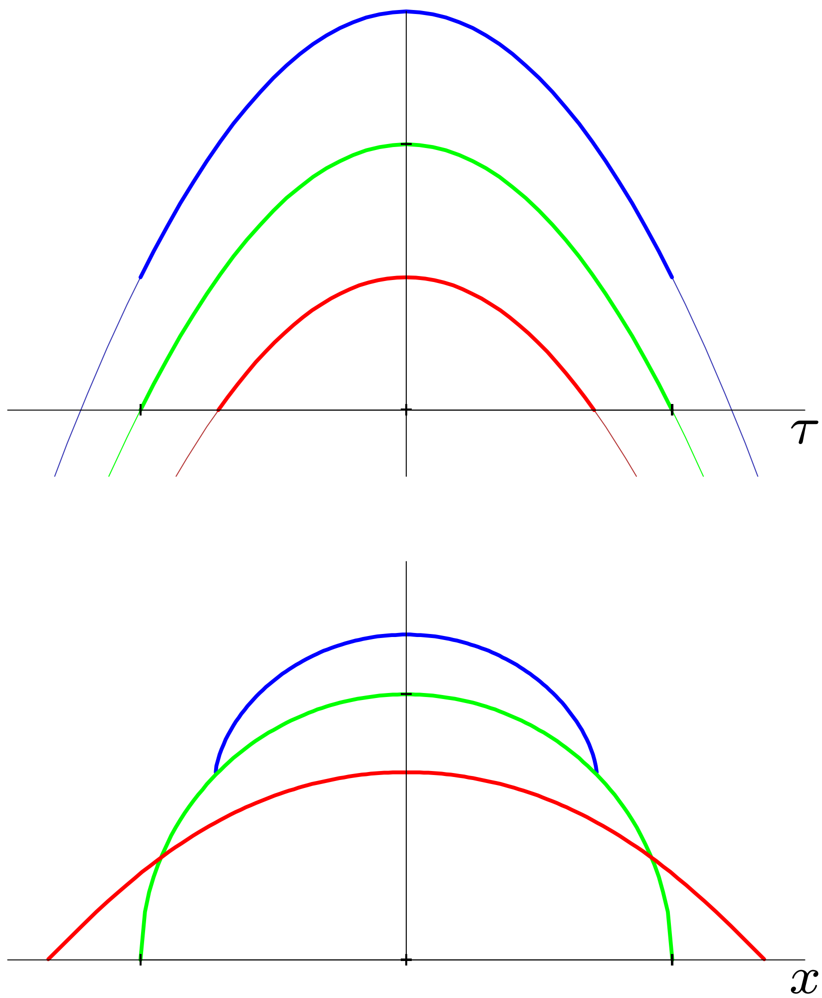
Constant curvature $\leftrightarrow$ quadratic polynomial

Affine functions:



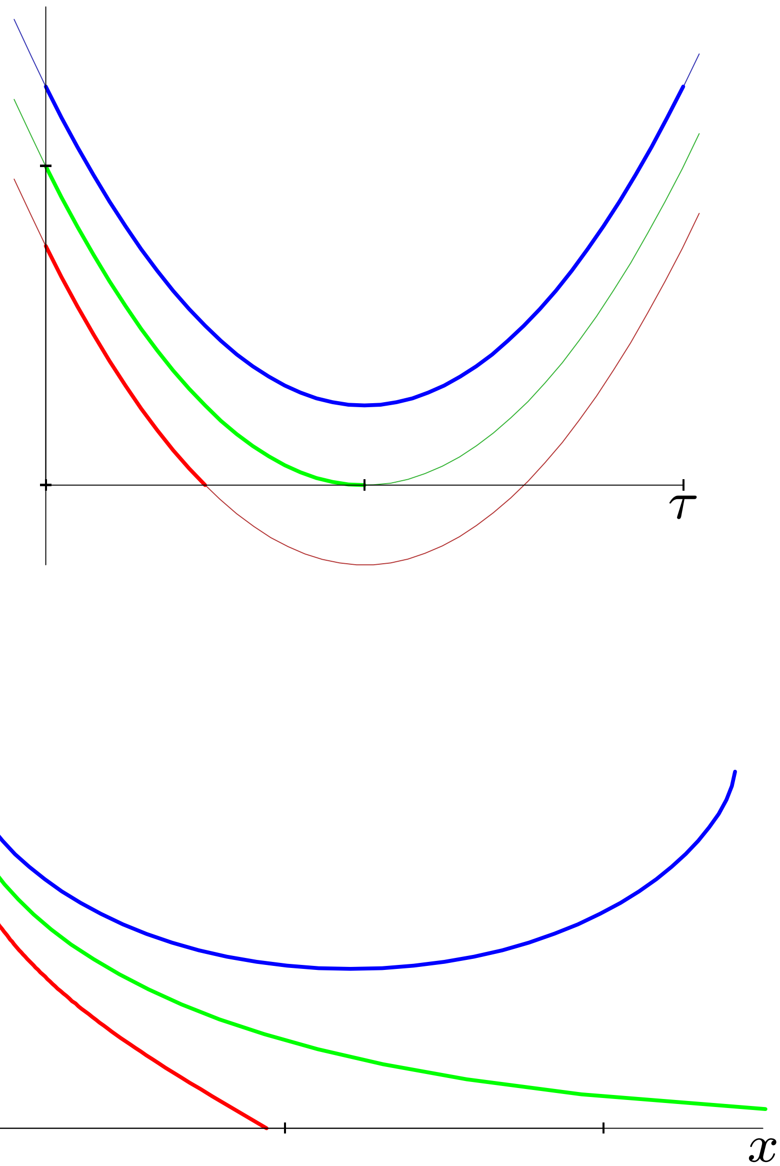


Concave quadratics: $\varphi(\tau) = a^2 - \tau^2$



Sphere is unique smooth, complete surface;
Some admit geodesics of length $> \pi$

Convex quadratics: $\varphi(\tau) = a^2 - 2\tau + \tau^2$



Pseudosphere has cusp end, $\varphi'(0) = 0$;
All admit extensions, many smooth