Musical, Physical, and Mathematical Intervals The 2010 Leonard Sulski Lecture College of the Holy Cross

Rick Miranda, Colorado State University

April 12, 2010

Outline

Vibrating Strings

When a string vibrates, its basic pitch
(the frequency of the sound wave generated)
is determined by

- the composition of the string (thickness, material, etc.)
- the tension of the string
- the length of the string.

Vibrating Strings

When a string vibrates, its basic pitch
(the frequency of the sound wave generated)
is determined by

- the composition of the string (thickness, material, etc.)
- the tension of the string
- the length of the string.

Ancient Scientists knew that Frequency and Length are *inversely* proportional:

$$\text{Frequency} = \frac{\text{(constant)}}{\text{Length}}$$

(Although they didn't really know much about Frequency...)

Pythagorean Intervals

The Pythagorean School refined this one step further.
They considered two notes together: Harmony

Pythagorean Intervals

The Pythagorean School refined this one step further.
They considered two notes together: Harmony
They noticed that the *most pleasing* harmonies
were produced by Frequencies
(actually, they used Lengths as the measure)
which were in ratios of small integers:

Pythagorean Intervals

The Pythagorean School refined this one step further.
They considered two notes together: Harmony
They noticed that the *most pleasing* harmonies
were produced by Frequencies
(actually, they used Lengths as the measure)
which were in ratios of small integers:

- Octave: (e.g. middle C to high C): 1 - to - 2
- Fifth: (e.g. C to G): 2 - to - 3
- Fourth: (e.g. C to F): 3 - to - 4
- Etc.

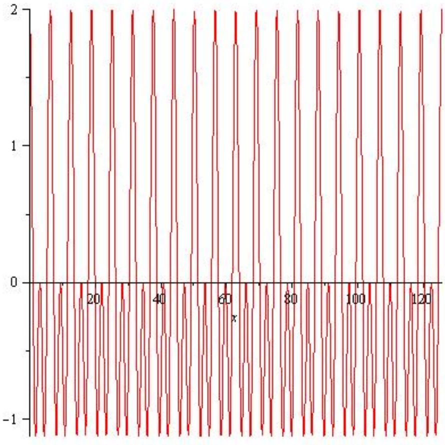
Pythagorean Intervals

The Pythagorean School refined this one step further.
They considered two notes together: Harmony
They noticed that the *most pleasing* harmonies
were produced by Frequencies
(actually, they used Lengths as the measure)
which were in ratios of small integers:

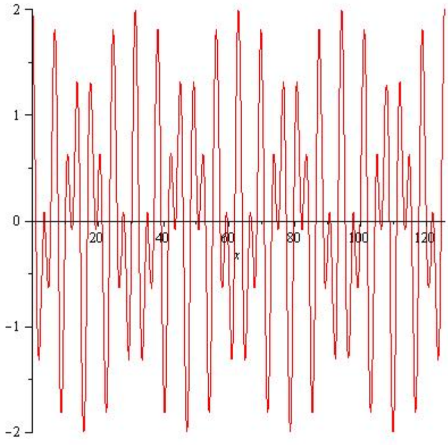
- Octave: (e.g. middle C to high C): 1 - to - 2
- Fifth: (e.g. C to G): 2 - to - 3
- Fourth: (e.g. C to F): 3 - to - 4
- Etc.

There is a "Resonance" reason for this:
the wave form produced by adding waves with these ratios
are simpler, less discordant (even visually)

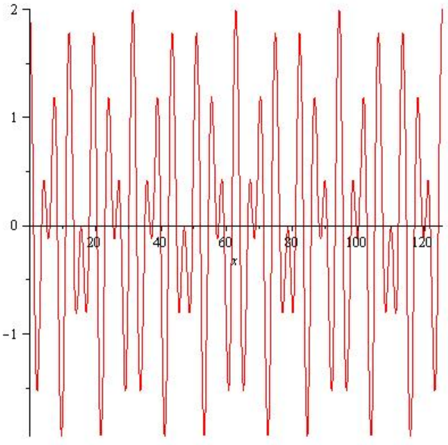
Octaves: Ratio = 2



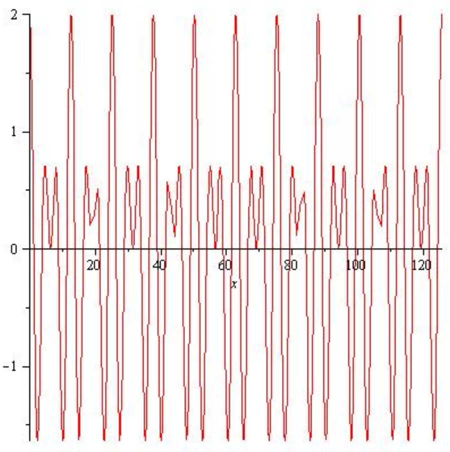
Dissonance: Ratio = 1.8



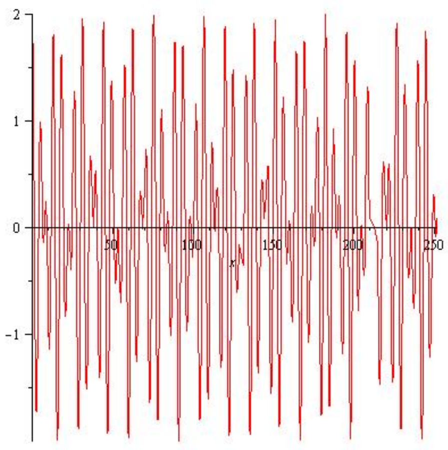
More Dissonance: Ratio = 1.6



Fifths: Ratio = 1.5



Fifths: Ratio $= \sqrt{2}$



Fifths and Fourths seem consistent, at least for a while:

$$\begin{array}{cccccc} F & G & C & F & G & C \\ \frac{3}{2} & \frac{4}{3} & 1 & \frac{3}{4} & \frac{2}{3} & \frac{1}{2} \end{array}$$

Fifths and Fourths seem consistent, at least for a while:

$$\begin{matrix} F & G & C & F & G & C \\ \frac{3}{2} & \frac{4}{3} & 1 & \frac{3}{4} & \frac{2}{3} & \frac{1}{2} \end{matrix}$$

- Octaves: F - to - F ratio $= \frac{3}{4}/\frac{3}{2} = \frac{1}{2}$.
- Also G - to - G ratio $= \frac{2}{3}/\frac{4}{3} = \frac{1}{2}$.
- Fifths: F - to - C ratio $= 1/\frac{3}{2} = \frac{2}{3}$;
- Also upper F - to - C ratio $= \frac{1}{2}/\frac{3}{4} = \frac{2}{3}$.
- Fourths: G - to - C ratio $= 1/\frac{4}{3} = \frac{1}{2}/\frac{2}{3} = \frac{3}{4}$.

More Notes To The Scale

This Pythagorean model works well for scales that only involve C's, F's, and G's.
Let's try to add a few more notes to the scale.

More Notes To The Scale

This Pythagorean model works well for scales that only involve C's, F's, and G's.

Let's try to add a few more notes to the scale.

A fifth above lower G is a D, and the Length should be

$$\frac{2}{3} * \frac{4}{3} = \frac{8}{9}.$$

More Notes To The Scale

This Pythagorean model works well for scales that only involve C's, F's, and G's.

Let's try to add a few more notes to the scale.

A fifth above lower G is a D, and the Length should be

$$\frac{2}{3} * \frac{4}{3} = \frac{8}{9}.$$

A fifth above that D is an A, and the Length should be

$$\frac{2}{3} * \frac{8}{9} = \frac{16}{27}.$$

More Notes To The Scale

This Pythagorean model works well for scales that only involve C's, F's, and G's.

Let's try to add a few more notes to the scale.

A fifth above lower G is a D, and the Length should be

$$\frac{2}{3} * \frac{4}{3} = \frac{8}{9}.$$

A fifth above that D is an A, and the Length should be

$$\frac{2}{3} * \frac{8}{9} = \frac{16}{27}.$$

A fourth below that A is an E, and the Length should be

$$\frac{4}{3} * \frac{16}{27} = \frac{64}{81}.$$

More Notes To The Scale

This Pythagorean model works well for scales that only involve C's, F's, and G's.

Let's try to add a few more notes to the scale.

A fifth above lower G is a D, and the Length should be

$$\frac{2}{3} * \frac{4}{3} = \frac{8}{9}.$$

A fifth above that D is an A, and the Length should be

$$\frac{2}{3} * \frac{8}{9} = \frac{16}{27}.$$

A fourth below that A is an E, and the Length should be

$$\frac{4}{3} * \frac{16}{27} = \frac{64}{81}.$$

A fifth above that E is an B, and the Length should be

$$\frac{2}{3} * \frac{64}{81} = \frac{128}{243}.$$

Il Diavolo In Musica

This gives the "white notes on the piano" scale:

C		D		E		F		G		A		B		C
1		$\frac{8}{9}$		$\frac{64}{81}$		$\frac{3}{4}$		$\frac{2}{3}$		$\frac{16}{27}$		$\frac{128}{243}$		$\frac{1}{2}$
	$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$	

Il Diavolo In Musica

This gives the "white notes on the piano" scale:

C		D		E		F		G		A		B		C
1		$\frac{8}{9}$		$\frac{64}{81}$		$\frac{3}{4}$		$\frac{2}{3}$		$\frac{16}{27}$		$\frac{128}{243}$		$\frac{1}{2}$
	$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$	

This leads to the scheme of:

- Whole Note $=$ ratio of $8/9$
- Half Note $=$ ratio of $243/256$

Il Diavolo In Musica

This gives the "white notes on the piano" scale:

C		D		E		F		G		A		B		C
1		$\frac{8}{9}$		$\frac{64}{81}$		$\frac{3}{4}$		$\frac{2}{3}$		$\frac{16}{27}$		$\frac{128}{243}$		$\frac{1}{2}$
	$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$	

This leads to the scheme of:

- Whole Note = ratio of $8/9$
- Half Note = ratio of $243/256$

And the Problem ("Il Diavolo") is that
Two Half Notes should equal a Whole note; but $(\frac{243}{256})^2 \neq \frac{8}{9}$!

Il Diavolo In Musica

This gives the "white notes on the piano" scale:

C		D		E		F		G		A		B		C
1		$\frac{8}{9}$		$\frac{64}{81}$		$\frac{3}{4}$		$\frac{2}{3}$		$\frac{16}{27}$		$\frac{128}{243}$		$\frac{1}{2}$
	$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{8}{9}$		$\frac{243}{256}$	

This leads to the scheme of:

- Whole Note $=$ ratio of $8/9$
- Half Note $=$ ratio of $243/256$

And the Problem ("Il Diavolo") is that
Two Half Notes should equal a Whole note; but $(\frac{243}{256})^2 \neq \frac{8}{9}$!
Close though:

$$(\frac{243}{256})^2 = .901016235 \text{ while } \frac{8}{9} = .888888888$$

About One Point Three Percent Off. "Pythagorean Comma"

12 Note Scales?

It gets worse if you try to make a full 12-note scale (including the 'black notes on the piano').

12 Note Scales?

It gets worse if you try to make a full 12-note scale (including the 'black notes on the piano').
Twelve Fifths (C - to - G)
should be the same as Seven Octaves.

12 Note Scales?

It gets worse if you try to make a full 12-note scale (including the 'black notes on the piano').
Twelve Fifths (C - to - G)
should be the same as Seven Octaves.
But $(2/3)^{12} \neq (1/2)^7$: This is equivalent to
$524288 = 2^{19} \neq 3^{12} = 531441$. (1.3% off...)

12 Note Scales?

It gets worse if you try to make a full 12-note scale (including the 'black notes on the piano').
Twelve Fifths (C - to - G)
should be the same as Seven Octaves.
But $(2/3)^{12} \neq (1/2)^7$: This is equivalent to
$524288 = 2^{19} \neq 3^{12} = 531441$. (1.3% off...)
Is there any way to recover from this?

12 Note Scales?

It gets worse if you try to make a full 12-note scale (including the 'black notes on the piano').
Twelve Fifths (C - to - G)
should be the same as Seven Octaves.
But $(2/3)^{12} \neq (1/2)^7$: This is equivalent to
$524288 = 2^{19} \neq 3^{12} = 531441$. (1.3% off...)
Is there any way to recover from this?
Not really.
No system of ratios enjoys the following properties:

- The ratio of all half notes (or all whole notes, or...) are the same
- The ratio of octaves is 1 - to - 2
- The ratio of fifths is 2 - to - 3.

Suppose you set the ratio of half note Frequencies to be a fixed number H_F.

Suppose you set the ratio of half note Frequencies to be a fixed number H_F.
There are 12 half-notes in an octave.
So you then need $(H_F)^{12} = 2$.

Suppose you set the ratio of half note Frequencies to be a fixed number H_F.
There are 12 half-notes in an octave.
So you then need $(H_F)^{12} = 2$.
This is a number: $H_F = \sqrt[12]{2} = 1.059463094\cdots$
The corresponding ratio of Lengths would then be

$$H_L = 1/H_F = 1/\sqrt[12]{2} = 1/1.059463094 = .943874313\cdots$$

Suppose you set the ratio of half note Frequencies to be a fixed number H_F.
There are 12 half-notes in an octave.
So you then need $(H_F)^{12} = 2$.
This is a number: $H_F = \sqrt[12]{2} = 1.059463094\cdots$
The corresponding ratio of Lengths would then be

$$H_L = 1/H_F = 1/\sqrt[12]{2} = 1/1.059463094 = .943874313\cdots$$

Compare this with the Pythagorean half-note length ratio of

$$\frac{243}{256} = .94921875\cdots$$

(These differ by about a half of one percent.)

Suppose you set the ratio of half note Frequencies to be a fixed number H_F.
There are 12 half-notes in an octave.
So you then need $(H_F)^{12} = 2$.
This is a number: $H_F = \sqrt[12]{2} = 1.059463094\cdots$
The corresponding ratio of Lengths would then be

$$H_L = 1/H_F = 1/\sqrt[12]{2} = 1/1.059463094 = .943874313\cdots$$

Compare this with the Pythagorean half-note length ratio of

$$\frac{243}{256} = .94921875\cdots$$

(These differ by about a half of one percent.)
Whole note ratios are then

$$L_F^2 = .890898\cdots \qquad (\frac{8}{9} = .888888\cdots)$$

These differ by about a fifth of one percent.

Compare the scales:

Note	C	D	E	F	G	A	B	C
Pyth.	1	.8889	.7901	.75	.6667	.5926	.5267	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	-0.226	-0.451	-0.113	-0.113	-0.338	-0.563	0

Minor Second and Major Third are the worst.

Compare the scales:

Note	C	D	E	F	G	A	B	C
Pyth.	1	.8889	.7901	.75	.6667	.5926	.5267	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	-0.226	-0.451	-0.113	-0.113	-0.338	-0.563	0

Minor Second and Major Third are the worst.

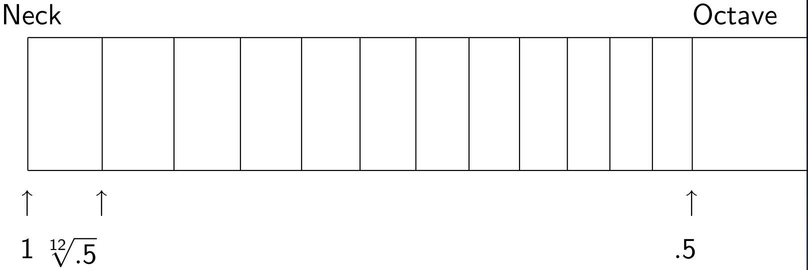
It is said that a musician's ear can tolerate about $\frac{1}{4} = 0.25$ percent

Compare the scales:

Note	C	D	E	F	G	A	B	C
Pyth.	1	.8889	.7901	.75	.6667	.5926	.5267	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	-0.226	-0.451	-0.113	-0.113	-0.338	-0.563	0

Minor Second and Major Third are the worst.
It is said that a musician's ear can tolerate about $\frac{1}{4} = 0.25$ percent before running screaming from the room.

Fretting A Guitar



Problem: There is **NO** geometric construction using a straight-edge and compass that will construct a length of $\sqrt[12]{.5}$.

Problem: There is **NO** geometric construction using a straight-edge and compass that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory, we discuss this in senior-level algebra courses.

Problem: There is **NO** geometric construction
using a straight-edge and compass
that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory,
we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube':
Given a cube, construct a cube of exactly twice the volume.

Problem: There is **NO** geometric construction
using a straight-edge and compass
that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory,
we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube':
Given a cube, construct a cube of exactly twice the volume.
Possible Solutions?

Problem: There is **NO** geometric construction
using a straight-edge and compass
that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory,
we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube':
Given a cube, construct a cube of exactly twice the volume.
Possible Solutions?

- Play the Violin instead

Problem: There is **NO** geometric construction
using a straight-edge and compass
that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory,
we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube':
Given a cube, construct a cube of exactly twice the volume.
Possible Solutions?

- Play the Violin instead
- Use a Computer

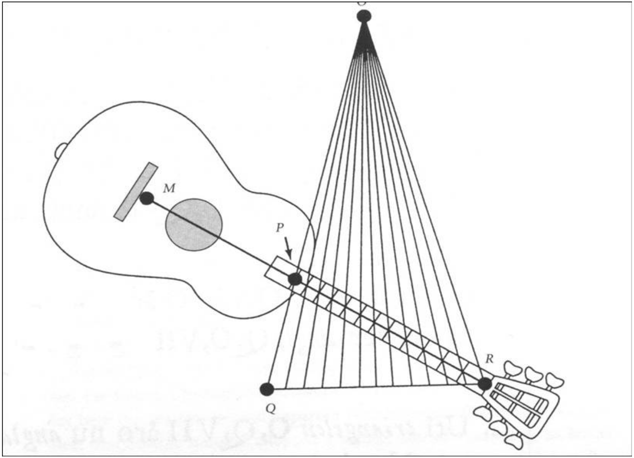
Problem: There is **NO** geometric construction using a straight-edge and compass that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory, we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube': Given a cube, construct a cube of exactly twice the volume.
Possible Solutions?

- Play the Violin instead
- Use a Computer (not available in the Renaissance)

Problem: There is **NO** geometric construction
using a straight-edge and compass
that will construct a length of $\sqrt[12]{.5}$.
This is a consequence of Galois Theory,
we discuss this in senior-level algebra courses.
It is equivalent to the problem of 'duplicating the cube':
Given a cube, construct a cube of exactly twice the volume.
Possible Solutions?

- Play the Violin instead
- Use a Computer (not available in the Renaissance)
- Approximate somehow

Strahle's Construction (exposed by Barbour 1957)



- Lay out a segment QR of length 12

- Lay out a segment QR of length 12
- Construct an isosceles triangle OQR with sides of length 24

- Lay out a segment QR of length 12
- Construct an isosceles triangle OQR with sides of length 24
- Fix the point P on OQ such that PQ has length 7

- Lay out a segment QR of length 12
- Construct an isosceles triangle OQR with sides of length 24
- Fix the point P on OQ such that PQ has length 7
- Draw the line RP and the point M on that line with MP = RP
  (the guitar, with neck at R, bridge at M, and octave at P)

- Lay out a segment QR of length 12
- Construct an isosceles triangle OQR with sides of length 24
- Fix the point P on OQ such that PQ has length 7
- Draw the line RP and the point M on that line with MP = RP
  (the guitar, with neck at R, bridge at M, and octave at P)
- Fret the guitar at the intersections of MR with the lines through O meeting QR at the 12 points dividing QR equally.

- Lay out a segment QR of length 12
- Construct an isosceles triangle OQR with sides of length 24
- Fix the point P on OQ such that PQ has length 7
- Draw the line RP and the point M on that line with MP = RP
  (the guitar, with neck at R, bridge at M, and octave at P)
- Fret the guitar at the intersections of MR with the lines through O meeting QR at the 12 points dividing QR equally.

Could this work?

The Mathematics of Projections

Suppose you have two lines L_1 and L_2 in the plane and a point O not on either line.

The Mathematics of Projections

Suppose you have two lines L_1 and L_2 in the plane and a point O not on either line.

Then one has a correspondence between the points of L_1 and L_2 given by "projection" from O.

The Mathematics of Projections

Suppose you have two lines L_1 and L_2 in the plane and a point O not on either line.
Then one has a correspondence between the points of L_1 and L_2 given by "projection" from O.
The projection $\pi : L_1 \to L_2$ is defined geometrically, but a formula for π can be obtained if one has coordinate systems on the two lines.

The Mathematics of Projections

Suppose you have two lines L_1 and L_2 in the plane and a point O not on either line.
Then one has a correspondence between the points of L_1 and L_2 given by "projection" from O.
The projection $\pi : L_1 \to L_2$ is defined geometrically, but a formula for π can be obtained if one has coordinate systems on the two lines.
Indeed, if x is a coordinate on L_1 and y is a coordinate on L_2 (with different origins, and different scales, allowed) then the mapping π will send a point on L_1 with coordinate x to a point on L_2 with coordinate $y = y(x)$; and this function **always** has the form

$$y(x) = \frac{a + bx}{c + dx}$$

for suitable constants a, b, c, and d.

For Strahle's construction, if you have a coordinate x on the segment QR which is 0 at R and 1 at Q, and a coordinate y on the guitar which is 0 at M and 1 at R, then the projection function is

$$y = \frac{17 - 5x}{17 + 7x}.$$

For Strahle's construction, if you have a coordinate x on the segment QR which is 0 at R and 1 at Q, and a coordinate y on the guitar which is 0 at M and 1 at R, then the projection function is

$$y = \frac{17 - 5x}{17 + 7x}.$$

This gives the lengths for the notes as:

Note	C	D	E	F	G	A	B	C
Strahle	1	.8899	.7931	.7490	.6680	.5955	.5302	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	-0.111	-0.075	0.027	0.085	0.15	0.098	0

For Strahle's construction, if you have a coordinate x on the segment QR which is 0 at R and 1 at Q, and a coordinate y on the guitar which is 0 at M and 1 at R, then the projection function is

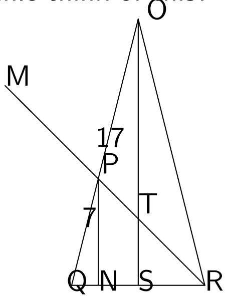
$$y = \frac{17 - 5x}{17 + 7x}.$$

This gives the lengths for the notes as:

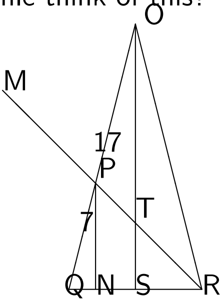
Note	C	D	E	F	G	A	B	C
Strahle	1	.8899	.7931	.7490	.6680	.5955	.5302	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	−0.111	−0.075	0.027	0.085	0.15	0.098	0

Pretty darn good!

- Why is $(17 - 5x)/(17 + 7x)$ so good?
- How did Strahle think of this?



- Why is $(17-5x)/(17+7x)$ so good?
- How did Strahle think of this?



- PQN similar to OQS; hence $|QN|/7 = |QS|/24$ or $|QN| = \frac{7}{24} * |QS| = \frac{7}{48} * |QR|$.
- Hence $|NR| = \frac{41}{48} * |QR|$; and $|SR|/|NR| = \frac{1/2}{41/48} = 24/41$.
- PNR similar to TSR; hence $|TR|/|SR| = |PR|/|NR|$
- $|PR| = |MR|/2$; Hence

$$|TR| = |PR| * (|SR|/|NR|) = (12/41) * |MR|.$$

- Therefore $|MT| = (29/41) * |MR|$.

The Most Accurate Projection

Suppose you look for a projection function

$$y(x) = \frac{a + bx}{c + dx}$$

which is gives the most accurate lengths for the notes.

The Most Accurate Projection

Suppose you look for a projection function

$$y(x) = \frac{a + bx}{c + dx}$$

which is gives the most accurate lengths for the notes. This means you'd want constants a, b, c, and d such that

$$\frac{a + bx}{c + dx} \approx (.5)^x$$

and your frets could then be placed by substituting $x = 0$, $1/12$, $2/12, \ldots, 11/12$, 1 into the linear fractional formula.

The Most Accurate Projection

Suppose you look for a projection function

$$y(x) = \frac{a + bx}{c + dx}$$

which is gives the most accurate lengths for the notes. This means you'd want constants a, b, c, and d such that

$$\frac{a + bx}{c + dx} \approx (.5)^x$$

and your frets could then be placed by substituting $x = 0$, $1/12$, $2/12, \ldots, 11/12$, 1 into the linear fractional formula. This appears to be four degrees of freedom, but is actually only three. (Only the ratios of the a,b,c,d count.)

The Most Accurate Projection

Suppose you look for a projection function

$$y(x) = \frac{a + bx}{c + dx}$$

which is gives the most accurate lengths for the notes. This means you'd want constants a, b, c, and d such that

$$\frac{a + bx}{c + dx} \approx (.5)^x$$

and your frets could then be placed by substituting $x = 0$, $1/12$, $2/12, \ldots, 11/12$, 1 into the linear fractional formula. This appears to be four degrees of freedom, but is actually only three. (Only the ratios of the a,b,c,d count.)
You would have to have

$$y(0) = 1 \quad \text{and} \quad y(1) = 1/2$$

in order to fix the neck and the octave exactly.

If you try to place the half-way note ('Tritone') exactly, you would then need

$$\frac{a+b/2}{c+d/2} = (.5)^{6/12} = \sqrt{.5} = 1/\sqrt{2}.$$

If you try to place the half-way note ('Tritone') exactly, you would then need

$$\frac{a + b/2}{c + d/2} = (.5)^{6/12} = \sqrt{.5} = 1/\sqrt{2}.$$

Solving these three equations for a, b, c, and d (and remembering that only the ratios count) leads to the **best approximate projection function**:

$$(.5)^x \approx \frac{(2 - \sqrt{2}) + (2\sqrt{2} - 3)x}{(2 - \sqrt{2}) + (3\sqrt{2} - 4)x}.$$

If you try to place the half-way note ('Tritone') exactly, you would then need

$$\frac{a+b/2}{c+d/2} = (.5)^{6/12} = \sqrt{.5} = 1/\sqrt{2}.$$

Solving these three equations for a, b, c, and d (and remembering that only the ratios count) leads to the **best approximate projection function**:

$$(.5)^x \approx \frac{(2-\sqrt{2}) + (2\sqrt{2}-3)x}{(2-\sqrt{2}) + (3\sqrt{2}-4)x}.$$

Yucchh!

If you try to place the half-way note ('Tritone') exactly, you would then need

$$\frac{a + b/2}{c + d/2} = (.5)^{6/12} = \sqrt{.5} = 1/\sqrt{2}.$$

Solving these three equations for a, b, c, and d (and remembering that only the ratios count) leads to the **best approximate projection function**:

$$(.5)^x \approx \frac{(2-\sqrt{2}) + (2\sqrt{2}-3)x}{(2-\sqrt{2}) + (3\sqrt{2}-4)x}.$$

Yucchh!
This is **NOT** what Strahle came up with, and it is not likely that a simple geometric construction like his would find this exact projection.

Continued Fraction Approximations of Numbers

Strahle's formula

$$y(x) = \frac{17 - 5x}{17 + 7x}$$

satisfies

$$y(0) = 1, y(1) = 1/2, \text{ but } y(1/2) = \frac{17 - 5/2}{17 + 7/2} = \frac{34 - 5}{34 + 7} = \frac{29}{41}.$$

Continued Fraction Approximations of Numbers

Strahle's formula

$$y(x) = \frac{17 - 5x}{17 + 7x}$$

satisfies

$$y(0) = 1, y(1) = 1/2, \text{ but } y(1/2) = \frac{17 - 5/2}{17 + 7/2} = \frac{34 - 5}{34 + 7} = \frac{29}{41}.$$

$$\frac{29}{41} \neq \sqrt{.5} = 1/\sqrt{2} \quad \text{since} \quad \frac{41}{29} \neq \sqrt{2}.$$

Continued Fraction Approximations of Numbers

Strahle's formula

$$y(x) = \frac{17 - 5x}{17 + 7x}$$

satisfies

$$y(0) = 1, y(1) = 1/2, \text{ but } y(1/2) = \frac{17 - 5/2}{17 + 7/2} = \frac{34 - 5}{34 + 7} = \frac{29}{41}.$$

$$\frac{29}{41} \neq \sqrt{.5} = 1/\sqrt{2} \quad \text{since} \quad \frac{41}{29} \neq \sqrt{2}.$$

Indeed, there is no rational number p/q such that $p/q = \sqrt{2}$; squaring both sides and multiplying by q^2 would give

$$p^2 = 2q^2$$

and this can't be true if p and q have no common factors.

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \qquad \text{"Pell's Equation"}$$

the closest one could get.

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \quad \text{"Pell's Equation"}$$

the closest one could get.
Note that

$$41^2 = 1681, \quad 29^2 = 841, \quad 2*29^2 = 1682, \quad 41^2 - 2*29^2 = -1$$

so $41/29$ is a **great** approximation to $\sqrt{2}$.

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \quad \text{"Pell's Equation"}$$

the closest one could get.
Note that

$$41^2 = 1681, \quad 29^2 = 841, \quad 2*29^2 = 1682, \quad 41^2 - 2*29^2 = -1$$

so $41/29$ is a **great** approximation to $\sqrt{2}$.
Indeed,

$$\frac{41}{29} = 1.413793103 \quad \text{and} \quad \sqrt{2} = 1.414213562$$

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \quad \text{"Pell's Equation"}$$

the closest one could get.
Note that

$$41^2 = 1681, \quad 29^2 = 841, \quad 2*29^2 = 1682, \quad 41^2 - 2*29^2 = -1$$

so $41/29$ is a **great** approximation to $\sqrt{2}$.
Indeed,

$$\frac{41}{29} = 1.413793103 \quad \text{and} \quad \sqrt{2} = 1.414213562$$

$$\frac{41}{29} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}}$$

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \quad \text{"Pell's Equation"}$$

the closest one could get.
Note that

$$41^2 = 1681, \quad 29^2 = 841, \quad 2*29^2 = 1682, \quad 41^2 - 2*29^2 = -1$$

so $41/29$ is a **great** approximation to $\sqrt{2}$.
Indeed,

$$\frac{41}{29} = 1.413793103 \quad \text{and} \quad \sqrt{2} = 1.414213562$$

$$\frac{41}{29} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}}$$

which is the truncation of the full continued fraction expansion of $\sqrt{2}$, and all truncations give all solutions to Pell's Equation, and the **best** rational approximations to $\sqrt{2}$.

But there **are** numbers such that

$$p^2 - 2q^2 = \pm 1, \quad \text{"Pell's Equation"}$$

the closest one could get.
Note that

$$41^2 = 1681, \quad 29^2 = 841, \quad 2*29^2 = 1682, \quad 41^2 - 2*29^2 = -1$$

so $41/29$ is a **great** approximation to $\sqrt{2}$.
Indeed,

$$\frac{41}{29} = 1.413793103 \quad \text{and} \quad \sqrt{2} = 1.414213562$$

$$\frac{41}{29} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}}$$

which is the truncation of the full continued fraction expansion of $\sqrt{2}$, and all truncations give all solutions to Pell's Equation, and the **best** rational approximations to $\sqrt{2}$.
There is no evidence at all that Strahle knew any of this!

Galilei's Approximation

Vincenzo Galilei:
the father of the famous astronomer Galileo Galilei

Galilei's Approximation

Vincenzo Galilei:
the father of the famous astronomer Galileo Galilei
He suggested using

$$\text{Half-note length ratio } = \frac{17}{18} = .944444444\cdots$$

Galilei's Approximation

Vincenzo Galilei:
the father of the famous astronomer Galileo Galilei
He suggested using

$$\text{Half-note length ratio } = \frac{17}{18} = .944444444\cdots$$

$$(\text{Pyth. } = \frac{243}{256} = .94921875\cdots \text{ and Equal } = .943874313\cdots)$$

Galilei's Approximation

Vincenzo Galilei:
the father of the famous astronomer Galileo Galilei
He suggested using

$$\text{Half-note length ratio } = \frac{17}{18} = .944444444\cdots$$

$$(\text{Pyth. } = \frac{243}{256} = .94921875\cdots \text{ and Equal } = .943874313\cdots)$$

(18/17 is the first continued fraction approximation to $\sqrt[12]{2}$.)

Galilei's Approximation

Vincenzo Galilei:
the father of the famous astronomer Galileo Galilei
He suggested using

$$\text{Half-note length ratio} = \frac{17}{18} = .944444444\cdots$$

$$(\text{Pyth.} = \frac{243}{256} = .94921875\cdots \text{ and Equal } = .943874313\cdots)$$

(18/17 is the first continued fraction approximation to $\sqrt[12]{2}$.)
If you compute, you find that

$$(\frac{17}{18})^{12} = 0.503636\cdots$$

so the Octave is off by .003636, a bit too short.

Vincenzo's solution was to just shorten the total string then, by exactly the amount to make this point (0.503636) halfway.

Vincenzo's solution was to just shorten the total string then, by exactly the amount to make this point (0.503636) halfway.
This point is $1 - .503636 = .496363734$ down the string, so you double it to get 0.992727468, cutting of 0.007272531 of the string.

Vincenzo's solution was to just shorten the total string then, by exactly the amount to make this point (0.503636) halfway.

This point is $1 - .503636 = .496363734$ down the string, so you double it to get 0.992727468, cutting of 0.007272531 of the string.

Mathematically, this makes the N^{th} note in the scale have length

$$\frac{(17/18)^N - .007272531}{0.992727468}$$

giving the lengths indicated below:

Note	C	D	E	F	G	A	B	C
Vincenzo	1	.8912	.7941	.7496	.6678	.5949	.5298	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	0.032	0.053	0.059	0.062	0.049	0.021	0

This is a fabulous approximation!

Vincenzo's solution was to just shorten the total string then, by exactly the amount to make this point (0.503636) halfway.

This point is $1 - .503636 = .496363734$ down the string, so you double it to get 0.992727468, cutting of 0.007272531 of the string.

Mathematically, this makes the N^{th} note in the scale have length

$$\frac{(17/18)^N - .007272531}{0.992727468}$$

giving the lengths indicated below:

Note	C	D	E	F	G	A	B	C
Vincenzo	1	.8912	.7941	.7496	.6678	.5949	.5298	.5
Equal	1	.8909	.7937	.7492	.6674	.5946	.5297	.5
Percent	0	0.032	0.053	0.059	0.062	0.049	0.021	0

This is a fabulous approximation!

His discovery that the pitch created by a string varied nonlinearly with the tension was one of the first non-linear physical laws discovered.

- J.M. Barbour: *A geometrical approximation to the roots of numbers*. American Mathematical Monthly, Vol. 64, No. 1 (1957), 1–9.
- V. Galilei: *Dialogo della musica antica e moderna*, Florence (1581), p. 49
- D.P. Strahle: *Nytt pafund, til at finna temperaturen i stamningen for thonerne pa claveret ock dylika instrumeter* Proceedings of the Swedish Academy (1743), Vol. IV, 281–291
- Ian Stewart