

Solution

There are n toothpicks and two players, A and B . Player A goes first. Each player can pick one or two toothpicks each time it is his/her turn.

Claim: If n is a multiple of 3 or a multiple of 3 plus 2 player A can force a win. If n is a multiple of 3 plus 1 player B can force a win. (In other words, if $n = 3k$ or $n = 3k + 2$ player A can force a win and if $n = 3k + 1$ player B can force a win.)

Proof: First we will show that if $n = 3k$ or $n = 3k + 2$, player A (first player) can always win. This can be done using mathematical induction, but it can also be explained to a high school student without explicitly referring to induction.

The winning strategy for player A is:

- (i) if the number of toothpicks on the table is a multiple of 3, pick two toothpicks.
- (ii) if the number of toothpicks on the table is a multiple of 3 plus 2, pick one toothpicks.

With this strategy, if the game started with $3k$ or $3k + 2$ toothpicks, there will never be a multiple of 3 plus 1 toothpicks when it is A 's turn.

Suppose $k = 1$. If there are 3 toothpicks. Player A picks 2 toothpicks. If there are 2 toothpicks, player A picks one toothpick. In either case, player B is forced to pick the last toothpick and player A wins.

Suppose $k > 1$ is arbitrary. If the game started with $3k$ toothpicks, player A picks two and there are $3k - 2 = 3(k - 1) + 1$ toothpicks left. If the game started with $3k + 2$ toothpicks, player A picks one toothpick and there are $3k + 1$ toothpicks left. In either case the number of toothpicks left is a multiple of 3 plus 1.

Case I: Player B picks one toothpick and there is now a multiple of 3 toothpicks left.

Case II: Player B picks two toothpicks and there is a multiple of 3 plus 2 toothpicks left.

Repeating this procedure, the game arrives to either two or three toothpicks when it is A 's turn and A can win.

If the game starts with $3k + 1$ toothpicks, there are two options for A :

Option 1: A picks one toothpick. There are $3k$ toothpicks left and B can win using the strategy described above.

Option 2: A picks two toothpicks. There are $3k - 1 = 3(k - 1) + 2$ toothpicks left and B can win using the strategy described above.