2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 9 – July 23, 2012

Traveling Salesman Problem/Pizza Cutting Problem

- Warm up exercise:
- Outline: "Discrete Functions"
  - Focus on "graphs" in a combinatorial sense
  - Motivated by practical/real worlds problems
  - Easy to explain, hard to solve
  - Functions connected to basic algebra
  - Functions don't come with nice formulas
  - The idea of an algorithm applied recursively arises here.
- The idea of a "graph"–This could be an IMP type big question, how might one "discover" these graphs? We'll try this here.
- The following comes after the IMP type exercise.
- **Finding a Hamiltonian circuit.**
  - We use the method of *trees*. A tree is graph that resembles an upside real tree. From the trunk, which is our starting point, Worcester, we begin adding branches so that no path descending the tree repeats a vertex, until the very end of a branch is Worcester. Let's try this:
    * It is connected. We can move along the edges of the tree from Worcester to get to any other school.
    * If we start by going to Fitchburg, it turns out we get stuck. If we then go to Amherst, we have to pass through Worcester to get to the east. If we then go to Lowell, we have to pass through Worcester on our way around to get to the west. The problem is that the path from Worcester to Fitchburg *disconnects* the graph. This is a position we don't want to be in.
    * If we start by going to Westfield State, we wind up circling clockwise until we get to Boston, where we have a choice and can go to Framingham or Mass Maritime. Either choice will produce a Hamiltonian circuit.
    * If we start by going to Framingham, then going to either Lowell or Boston will disconnect the graph as above. So we have to go to Bridgewater, then Dartmouth (why not Mass Maritime). This will be the reverse of one of the circuits we found above.
    * See the accompanying tree.

    - In the end there are only two Hamiltonian circuits. One has a total distance of 429.4 miles, the other 422.8 miles.
    - To summarize, we can find Hamiltonian circuits by building a tree and using the rule of do not take an edge which disconnects or cuts the graph into two pieces.

- **Where are the functions?**
  - The weights are a function on the collection of edges.
  - The total weight function is a function on the collection of paths or circuits or Hamiltonian circuits.
  - The number of distinct Hamiltonian circuits is a function of the graph!
  - All are functions of discrete variables. The counting function takes only discrete values.

  Comment.

- **Graph Coloring**: (If needed.) Chromatic number of graph. The smallest number of colors needed for a proper coloring of a graph. The chromatic polynomial counts the number of such colorings. Plug in the number of colors, out comes the number of colorings!

- **Pizza Cutting Problem**
  - Start with a round pizza in the plane. Cuts are straight line through the pizza. Find the maximum number of pizza pieces one can obtain with n cuts.
  - First look at what happens with 3 cuts.
  - To obtain the maximum number of pieces, the following must happen.

    (i) Any two lines (cuts) must intersect

    (ii) No more that two lines (cuts) can intersect in the same point.
  - Consider the function $P(n)$, which gives the *maximum* number of pizza pieces obtained with n cuts. Then,

    $$P(n+1) = P(n) + n + 1.$$

    This is an example of a function defines through a recurrence.

    To obtain an exact formula for $P(n)$, we had to show first that the sum of the first n whole numbers equals $\frac{n(n+1)}{2}$. We obtained

    $$P(n) = \frac{n^2 + n + 2}{2}.$$

- We proved this formula again, in a different way. We viewed the pizza and the cuts as a graph. The vertices are intersections of cuts or intersections of a cut and crust. The intersections of two cuts forms an *interior vertex*, while the intersection of a cut with crust forms a *boundary vertex*. For each cut, the boundary vertex that is "higher" is considered the *upper boundary vertex*. Go to each piece of pizza and mark the highest vertex with a small line segment (going into the piece of pizza). Then each interior vertex and each upper boundary vertex determines exactly one piece of pizza, except the highest upper boundary vertex, which determines two pieces of pizza. Then, when using n cuts,

  $$\text{max. \# of pizza pieces} = 1 + \text{\# upper boundary vertices} + \text{\# interior vertices}$$

  $$\text{max. \# of pizza pieces} = 1 + \text{\# of cuts} + \text{\# of pairs of cuts}$$

  $$\text{max. \# of pizza pieces} = 1 \; + n + n \text{ "choose" } 2$$

  If we write $\binom{n}{k}$ for n "choose" k, we have

  $$\text{max. \# of pizza pieces} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}.$$

- This can be generalized to give a formula for the maximum number of pieces of grapefruit obtained when we cut the grapefruit with n planes.

  $$\text{max. \# of grapefruit pieces} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$