2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 9 – July 23, 2012

Summary of Graphs:

Along the way, you have used several concepts that arise in the area of mathematics known as *discrete mathematics* or *combinatorics*. Here is a list of them:

- Going from Amherst to the Mass Maritime academy is the equivalent of moving along a collection of edges in order from one vertex to another so that the vertex at the end of one edge is the vertex of the beginning of the next. This is called a *path*.
- Starting at Worcester and ending at Worcester gives a path that starts at one vertex and ends at the same vertex. This is called a *circuit* or *cycle*.
- A route that allows you to visit every school is called a *Hamiltonian path* (if it has a different start and end points) or a *Hamiltonian circuit* (if it has the same starting and ending point). These are named after the 19th century British physicist, astronomer, and mathematician William Rowan Hamilton. Among other things, he is known for inventing his own system of numbers (can you imagine that!), called quarternions, which are important in mathematics and physics.

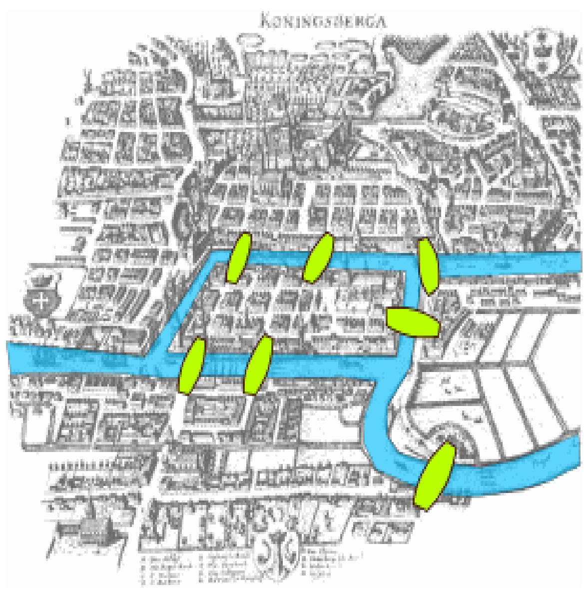

Figure 1: The bridges of Konigsberg, Prussia. (Wikipedia, entry for Leonhard Euler.)

- A route that allows you to drive every road between schools once is called an *Eulerian path* (if it has a different starting and ending points) or a *Eulerian circuit* (if it has the same start and end point). These are named after the 18th century German mathematician and physicist Leonhard Euler. Among many, many accomplishments, "In 1736, Euler solved the problem known as the Seven Bridges of Königsberg.[31] The city of Königsberg, Prussia was set on the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem is to decide whether it is possible to follow a path that crosses each bridge exactly once and returns to the starting point. It is not possible: there is no Eulerian circuit. This solution is considered to be the first theorem of graph theory, specifically of planar graph theory." (Wikipedia: Entry for Leonhard Euler.)

- In answering Xavier's and Sofjola's question you likely wound up thinking about how many roads there were in and out of a city, which corresponds to the number of edges that share a vertex. This is called the *valence* (or degree) of the vertex. In our graph, the vertex for Amherst has valence 2, for Fitchburg has valence 3, and for Worcester has valence 4. How did valence play a role in your answer?

- In answering Yaw and Tashara's question, choosing a path or circuit that produced a minumum of some quantity is an example of the *traveling salesman problem*. This problem involves a *weighted graph*. These are graphs with numbers or *weights* assigned to each edge. These might be a distance, as in our case, a time (which we did not use), or a cost, say the cost of gasoline in our case. The *weight* of path or circuit is the sum of the weights of the edges of the path or circuit. The goal is to find a path or circuit of smallest weight.

  Since we are working with a finite number of vertices and edges, one could list every possible Hamiltonian circuit and just add up the weights for each. Unfortunately, this number can be very large, so that a solution that does not require listing all Hamiltonian circuits is very desirable. Again unfortunately, this is not an easy problem to solve.

  A pictorial history in graphs of the traveling salesman problem can be found here:
  `http://www.tsp.gatech.edu/history/pictorial/dfj.html`.
  The parent site has much good information on the traveling salesman problem:
  `http://www.tsp.gatech.edu/index.html`

  The problem was first posed by William Rowan Hamilton. In fact, he invented a game (1857) called the *icosonian* game that involved this. (See attached.) The name "traveling salesman" was coined by the American mathematician Hassler Whitney in the 1930s.